A junction condition by specified homogenization and application to traffic lights

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Joint work with **Cyril Imbert** (CNRS, ENS - Paris) and **Régis Monneau** (ENPC)

Hamilton-Jacobi Equations : New trends and applications
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$$\begin{cases} u_t^\varepsilon + H\left(\frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0 & \text{for } (t,x) \in (0,T) \times \mathbb{R} \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbb{R} \end{cases}$$

with

$$H(x,p) = \begin{cases} H_R(p) & \text{if } x \geq 1 \\ H_L(p) & \text{if } x \leq -1 \\ \text{``regular''} & \text{if } x \in (-1,1), \end{cases}$$

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H **satisfies the following assumptions...**

(A0) **Continuity**: $H : \mathbb{R}^3 \to \mathbb{R}$ is continuous.

(A1) **Time periodicity**: for all $k \in \mathbb{Z}$ and $(t, x, p) \in \mathbb{R}^3$,

$$H(t + k, x, p) = H(t, x, p).$$

(A2) **Uniform modulus of continuity in time**: there exists a modulus of continuity ω such that for all $t, s, x, p \in \mathbb{R}$,

$$H(t, x, p) - H(s, x, p) \leq \omega(|t - s| \,(1 + \max(H(s, x, p), 0))).$$

(A3) **Uniform coercivity**: $\lim_{|q| \to +\infty} H(t, x, q) = +\infty$ uniformly w.r.t. (t, x).

(A4) **Quasi-convexity of H for large x's**: there exists some $\rho_0 > 0$ such that for all $x \in \mathbb{R} \setminus (-\rho_0, \rho_0)$, there exists a continuous map $t \mapsto p^0(t, x)$ such that

$$\begin{cases} H(t, x, \cdot) & \text{is non-increasing in} \quad (-\infty, p^0(t, x)), \\ H(t, x, \cdot) & \text{is non-decreasing in} \quad (p^0(t, x), +\infty). \end{cases}$$

(A5) **Left and right Hamiltonians**: there exist two Hamiltonians $H_\alpha(t, x, p)$, $\alpha = L, R$, such that

$$\begin{cases} H(t, x + k, p) - H_L(t, x, p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to -\infty \\ H(t, x + k, p) - H_R(t, x, p) \to 0 & \text{as} \quad \mathbb{Z} \ni k \to +\infty \end{cases}$$

uniformly with respect to $(t, x, p) \in [0, 1]^2 \times \mathbb{R}$, and for all $k, j \in \mathbb{Z}$, $(t, x, p) \in \mathbb{R}^3$ and $\alpha \in \{L, R\}$,

$$H_\alpha(t + k, x + j, p) = H_\alpha(t, x, p).$$

Examples

(i)

$$H(t,x,p) = \sqrt{|p|} + \sin(2\pi t)\,(1-|x|)^+$$
$$H_\alpha(p) = \sqrt{|p|} \qquad \alpha = L,\ R.$$

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(ii)

$$H(t,x,p) = |p| + \sin(2\pi t) + \tanh(x)\sin(2\pi x)$$
$$H_L(t,x,p) = |p| + \sin(2\pi t) - \sin(2\pi x)$$
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(iii)

$$H(x,t,p) = H_1(p) + f(t,x)$$

with
H_1 continuous, convex and coercive,
f continuous, $f(t+1,x) = f(t,x)$ and $\lim_{|x|\to+\infty} f(t,x) = 0$ uniformly with respect to $t \in \mathbb{R}$.

$$H_\alpha(p) = H_1(p) \qquad \alpha = L,\ R.$$

In order to ensure that the effective Hamiltonians $\overline{H}_L$, $\overline{H}_R$ are quasi-convex we need to impose additional assumptions:

(B-i) **Quasi-convexity of the left and right Hamiltonians**: for each $\alpha = L, R$, H_α does not depend on time and there exists p^0_α (independent on (t,x)) such that

$$\begin{cases} H_\alpha(x,\cdot) & \text{is non-increasing on} \quad (-\infty, p^0_\alpha), \\ H_\alpha(x,\cdot) & \text{is non-decreasing on} \quad (p^0_\alpha, +\infty). \end{cases}$$

(B-ii) **Convexity of the left and right Hamiltonians**: for each $\alpha = L, R$, and for all $(t,x) \in \mathbb{R} \times \mathbb{R}$, the map $p \mapsto H_\alpha(t,x,p)$ is convex.

THEOREM [G. - Imbert - Monneau, Analysis & PDE (2015)]

Assume **(A0)-(A5)** and either **(B-i)** or **(B-ii)**. Assume that the initial datum u_0 is Lipschitz continuous and for $\varepsilon > 0$, let u^ε be the solution of (CP). Then u^ε converges locally uniformly to the unique flux-limited solution u^0 of

$$\begin{cases} u^0_t + \overline{H}_L(u^0_x) = 0, & t > 0,\, x < 0, \\ u^0_t + \overline{H}_R(u^0_x) = 0, & t > 0,\, x > 0, \\ u^0_t + F_{\bar{A}}(u^0_x(t,0^-), u^0_x(t,0^+)) = 0, & t > 0,\, x = 0, \\ u^0(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

...the slopes of the limit solution at the origin are characterized by the effective flux limiter $\bar{A}$...

Some references

Homogenization

- Lions - Papanicolau - Varadhan (1986, unpublished)
- Evans (1989-1992, Proc. Roy. Soc. Edinburgh)

Specified homogenization

- Lions (2013-2014, Collège di France)

Time dependent Hamiltonians

- Barles - Souganidis (2000, JMA)
- Bernard - Roquejoffre (2001, CPDE) Large time behaviour
- Focardel - Imbert - Monneau (2009, JDE), (2009, DCDS), (2012, Trans. Amer. Math. Soc.) Homogenization, dislocation dynamics

Hamilton-Jacobi equations on networks and optimal control

- Achdou - Camilli - Cutrì - Tchou (2013, NoDEA)
- Achdou - Tchou (2015, CPDE)
- Barles - Briani - Chasseigne (2013, COCV)
- Imbert - Monneau-Zidani (2013, COCV)
- Imbert - Monneau (2016, preprint)

Hamilton-Jacobi equations with discontinuous source terms

- Giga - Hamamuki (2013, CPDE)

The homogenized left and right Hamiltonians are classically determined by the study of some "cell problems".

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Proposition

Assume **(A0)-(A5)**. Then for every $p \in \mathbb{R}$ and $\alpha = L, R$ there exists a unique $\lambda \in \mathbb{R}$ such that there exists a bounded (discontinuous) viscosity solution v^α of

$$\begin{cases} v_t^\alpha + H_\alpha(t, x, p + v_x^\alpha) = \lambda \quad \text{in} \quad \mathbb{R} \times \mathbb{R}, \\ v^\alpha \text{ is } \mathbb{Z}^2\text{-periodic.} \end{cases}$$

If $\overline{H}_\alpha(p)$ denotes such a λ, then the map $p \mapsto \overline{H}_\alpha(p)$ is continuous.

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Remark

If H_α does not depend on t, then it is possible to construct a corrector which does not depend on time either.

1 - Coercivity: $\lim\limits_{|p|\to+\infty} \overline{H}_\alpha(p) = +\infty$

Further properties of $\overline{H}_\alpha$

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Sketch of the proof. Let v_p be solutions of the cell problem

$$\begin{cases} (v_p)_t + H_\alpha(t,x,p+(v_p)_x) = \overline{H}_\alpha(p) \quad \text{in} \quad \mathbb{R}\times\mathbb{R}, \\ v_p \text{ is } \mathbb{Z}^2\text{-periodic} \end{cases}$$

and set

$$u_p(t,x) = v_p(t,x) + px - t\overline{H}_\alpha(p)\,.$$

Similarly we define

$$u_q(t,x) = v_q(t,x) + qx - t\overline{H}_\alpha(q)\,.$$

Step 1: *u_p and u_q are locally Lipschitz continuous.*

We have almost everywhere:

$$\begin{cases} (u_p)_t + H_\alpha(t, x, (u_p)_x) = 0, \\ (u_q)_t + H_\alpha(t, x, (u_q)_x) = 0. \end{cases}$$

For $\mu \in [0, 1]$, let

$$\bar{u} = \mu u_p + (1 - \mu) u_q.$$

By convexity, we get almost everywhere

$$\bar{u}_t + H_\alpha(t, x, \bar{u}_x) \leq 0.$$

For $P = (t, x)$, we define a mollifier $\rho_\delta(P) = \delta^{-2}\rho(\delta^{-1}P)$ and set

$$\bar{u}_\delta = \bar{u} \star \rho_\delta \to \bar{u} \quad \text{locally uniformly as } \delta \to 0.$$

Then by convexity, we get with $Q = (s, y)$:

$$(\bar{u}_\delta)_t + H_\alpha(P, (\bar{u}_\delta)_x) \leq \int dQ \ \left\{ H_\alpha(P, \bar{u}_x(Q)) - H_\alpha(Q, \bar{u}_x(Q)\right\} \rho_\delta(P - Q)$$
$$\to 0 \quad \text{as } \delta \to 0.$$

By stability of viscosity sub-solutions we deduce that

$$\bar{u}_t + H_\alpha(t, x, \bar{u}_x) \leq 0$$

in the viscosity sense.
Moreover, for $z = \mu p + (1-\mu)q$, if v_z is a $\mathbb{Z}^2$-periodic solution of the cell problem and $u_z(t,x) = v_z(t,x) + zx - t\overline{H}_\alpha(z)$ then

$$(u_z)_t + H_\alpha(t, x, (u_z)_x) = 0 \quad \text{in} \quad \mathbb{R} \times \mathbb{R},$$

and

$$\bar{u}(0,x) \leq u_z(0,x) + C \quad \text{in } \mathbb{R},$$

for C large enough.
Then the comparison principle implies that

$$t\overline{H}_\alpha(z) \leq t\left(\mu\overline{H}_\alpha(p) + (1-\mu)\overline{H}_\alpha(q)\right) + C$$

and

$$\overline{H}_\alpha(z) \leq \mu\overline{H}_\alpha(p) + (1-\mu)\overline{H}_\alpha(q)$$

for $t \to +\infty$.

Step 2: u_p *and* u_q *are continuous.*

For $\nu > 0$ and $z = p, q$, the functions

$$u_z^\nu(t,x) := \sup_{s\in\mathbb{R}} \left(u_z(s,x) - \frac{(t-s)^2}{2\nu} \right)$$

are **Lipschitz continuous** and satisfy the inequality

$$u_t^\nu + H(t,x,u_x^\nu) \leq o_\nu(1) \quad \text{a.e.}$$

where $o_\nu(1)$ is locally uniform with respect to (t,x).

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where $o_\nu(1)$ is locally uniform with respect to (t,x).
The *convex combination*

$$\bar{u}^\nu := \mu u_p^\nu + (1-\mu) u_q^\nu$$

is a viscosity subsolution of

$$(\bar{u}^\nu)_t + H_\alpha(t,x,(\bar{u}^\nu)_x) \leq o_\nu(1).$$

In the limit $\nu \to 0$, we recover (by stability of subsolutions) that $\bar{u}$ is a viscosity solution of

$$\bar{u}_t + H_\alpha(t,x,\bar{u}_x) \leq 0$$

and we conclude as in *Step 1*.

Step 3: general case.

We replace u_z, for $z = p, q$, by $\tilde{u}_z$ which is the continuous solution to the Cauchy problem

$$\begin{cases} (\tilde{u}_z)_t + H_\alpha(t, x, (\tilde{u}_z)_x) = 0, & \text{for} \quad (t, x) \in (0, +\infty) \times \mathbb{R} \\ \tilde{u}_z(0, x) = zx. \end{cases}$$

Then

$$|\tilde{u}_z - u_z| \leq C$$

and the *convex combination*

$$\tilde{u} = \mu \tilde{u}_p - (1 - \mu)\tilde{u}_z$$

is a viscosity subsolution, by *Step 2*, of

$$\tilde{u}_t + H(t, x, \tilde{u}_x) = 0.$$

The comparison principle yields the conclusion

$$\overline{H}_\alpha(\mu p + (1 - \mu) q \leq \mu \overline{H}_\alpha(p) + (1 - \mu)\overline{H}_\alpha(q). \quad \square$$

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Sketch of the proof. We first assume that H_α satisfies

$$\begin{cases} H_\alpha \in C^2, \\ D^2_{pp} H_\alpha(x, p^0_\alpha) > 0, \\ D_p H_\alpha(x,p) < 0 & \text{for} \quad p \in (-\infty, p^0_\alpha), \\ D_p H_\alpha(x,p) > 0 & \text{for} \quad p \in (p^0_\alpha, +\infty), \\ H_\alpha(x,p) \to +\infty & \text{as} \quad |p| \to +\infty \quad \text{uniformly w.r.t. } x \in \mathbb{R}. \end{cases}$$

From non-convex to convex H ...

There exists a convex function $\gamma \in C^2(\mathbb{R})$ s.t. $\gamma' \geq \delta_0 > 0$ and

$$D^2_{pp}(\gamma \circ H_\alpha) > 0$$

For $\lambda = \overline{\gamma \circ H_\alpha}(p)$ we can construct a *time independent corrector* of

$$\gamma \circ H_\alpha(x, p + v_x) = \overline{\gamma \circ H_\alpha}(p),$$

showing that

$$\overline{H}_\alpha = \gamma^{-1} \circ \overline{\gamma \circ H_\alpha}.$$

Since $\overline{\gamma \circ H_\alpha}$ is coercive and convex we deduce that

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$$\overline{H}_\alpha \text{ *is quasiconvex*.}$$

In the general case, for all $\varepsilon > 0$, there exists $H_\alpha^\varepsilon \in C^2$ such that

$$\begin{cases} (D_{pp}^2 H_\alpha^\varepsilon)(x, p_\alpha^0) > 0 \\ D_p H_\alpha^\varepsilon(x,p) < 0 \quad \text{for} \quad p \in (-\infty, p_\alpha^0), \\ D_p H_\alpha^\varepsilon(x,p) > 0 \quad \text{for} \quad p \in (p_\alpha^0, +\infty), \\ |H_\alpha^\varepsilon - H_\alpha| < \varepsilon. \end{cases}$$

Taking into account that $\overline{H}_\alpha(p) = \lim_{\varepsilon \to 0} \overline{H}_\alpha^\varepsilon(p)$ and that $\overline{H}_\alpha^\varepsilon$ is quasiconvex, then so is $\overline{H}_\alpha$.

Truncated cell problems

Problem: find $\lambda_\rho \in \mathbb{R}$ and w such that

$$\begin{cases} w_t + H(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times (-\rho,\rho), \\ w_t + H^-(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{-\rho\}, \\ w_t + H^+(t,x,w_x) = \lambda_\rho, & (t,x) \in \mathbb{R} \times \{\rho\}, \\ w \text{ is } 1\text{-periodic w.r.t. } t. \end{cases} \tag{TCP}$$

- we borrow here an idea from [Achdou and Tchou, CPDE (2016)] by truncating the domain and by considering correctors in $[-\rho,\rho]$ with $\rho \to +\infty$.

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Proposition - Correctors on truncated domains

There exists a unique $\lambda_\rho \in \mathbb{R}$ such that there exists a solution $w = w^\rho$ of (TCP). Moreover, there exists a constant $C > 0$ independent of $\rho \in (\rho_0, +\infty)$ and a function $m^\rho \colon [-\rho,\rho] \to \mathbb{R}$ such that

$$\begin{cases} |\lambda_\rho| \le C, \\ |m^\rho(x) - m^\rho(y)| \le C\,|x-y| & \text{for } x,y \in [-\rho,\rho], \\ |w^\rho(t,x) - m^\rho(x)| \le C & \text{for } (t,x) \in \mathbb{R} \times [-\rho,\rho]. \end{cases}$$

Proof.

Perron's method $\rightarrow$ discontinuous viscosity solution of

$$\begin{cases} \delta w^\delta + w^\delta_t + H(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times (-\rho,\rho)\,, \\ \delta w^\delta + w^\delta_t + H^-(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times \{-\rho\}\,, \\ \delta w^\delta + w^\delta_t + H^+(t,x,w^\delta_x) = 0, & (t,x) \in \mathbb{R} \times \{\rho\}\,, \\ w^\delta \text{ is } 1\text{-periodic w.r.t. } t. \end{cases}$$

satisfying

$$|w^\delta| \leq \frac{C}{\delta} \quad \text{with} \quad C = \sup_{(t,x)\in\mathbb{R}^2} |H(t,x,0)|.$$

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satisfying

$$|w^\delta| \le \frac{C}{\delta} \quad \text{with} \quad C = \sup_{(t,x)\in\mathbb{R}^2} |H(t,x,0)|.$$

Then there exists $\delta_n \to 0$ such that

$$\delta_n w^{\delta_n}(0,0) \to -\lambda_\rho \quad \text{as } n \to +\infty$$

and

$$|\lambda_\rho| \le C.$$

The function

$$m^\delta(x) = \sup_{t\in\mathbb{R}} (w^\delta)^*(t,x)$$

is a viscosity solution (for some function $t(x)$) of

$$H(t(x), x, m^\delta_x) \le C, \quad x \in (-\rho, \rho).$$

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- *Coercivity of H*:

$$|m^\delta_x| \le C \quad \text{and} \quad w^\delta_t \le C.$$

$$m^{\delta_n} - m^{\delta_n}(0) \to m^\rho \quad \text{locally uniformly as } n \to +\infty$$

$$|m^\rho(x) - m^\rho(y)| \le C\,|x-y|$$

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- *Comparison principle*: for all $t \in \mathbb{R}$, $x \in (-\rho, \rho)$ and $h \ge 0$,

$$w^\delta(t+h, x) \le w^\delta(t,x) + Ch.$$

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- *Comparison principle*: for all $t \in \mathbb{R}$, $x \in (-\rho,\rho)$ and $h \ge 0$,

$$w^\delta(t+h, x) \le w^\delta(t,x) + Ch.$$

- *Time periodicity of w^δ*: for $t \in \mathbb{R}$ and $x \in (-\rho,\rho)$,

$$|w^\delta(t,x) - m^\delta(x)| \le C \quad \text{and} \quad |w^\delta(t,x) - w^\delta(0,0)| \le C.$$

We then consider

$$\overline{w} = \limsup_{n\to+\infty} {}^*(w^{\delta_n} - w^{\delta_n}(0,0)) < +\infty$$

and

$$\underline{w} = \liminf_{n\to+\infty} {}_*(w^{\delta_n} - w^{\delta_n}(0,0)) < +\infty.$$

From the above estimates

$$|\overline{w} - m^\rho| \leq C \quad \text{and} \quad |\underline{w} - m^\rho| \leq C.$$

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$$\overline{w} - 2C \leq \underline{w}.$$

- *Perron's method:* solution of (TCP)

$$\overline{w} - 2C \leq w^\rho \leq \underline{w}\,, \quad |w^\rho(t,x) - m^\rho(x)| \leq C\,. \quad \square$$

The effective flux limiter $\bar{A}$

$$u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0 \xrightarrow{\varepsilon \to 0} \begin{cases} u_t^0 + \overline{H}_L(u_x^0) = 0, & t > 0,\ x < 0, \\ u_t^0 + \overline{H}_R(u_x^0) = 0, & t > 0,\ x > 0, \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0, & t > 0,\ x = 0. \end{cases}$$

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The effective junction function $F_{\bar{A}}$ is defined by

$$F_{\bar{A}}(p_L, p_R) := \max(\bar{A}, \overline{H}_L^+(p_L), \overline{H}_R^-(p_R))$$

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Theorem - Definition

- The map $\rho \mapsto \lambda_\rho$ is nondecreasing, bounded and $\bar{A} = \lim_{\rho\to+\infty} \lambda_\rho$.

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Theorem - Definition

- The map $\rho \mapsto \lambda_\rho$ is nondecreasing, bounded and $\bar{A} = \lim_{\rho \to +\infty} \lambda_\rho$.
- Consider the problem

$$\begin{cases} w_t + H(t, x, w_x) = \lambda, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases} \qquad (*)$$

The set $E = \{\lambda \in \mathbb{R} : \exists w \text{ sub-solution of } (*)\}$ is nonempty and bounded from below. Moreover $\bar{A} = \inf E$

(i) General properties

There exists a solution w of

$$\begin{cases} w_t + H(t,x,w_x) = \bar{A}, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ w \text{ is 1-periodic w.r.t. } t. \end{cases}$$

such that for all $(t,x) \in \mathbb{R} \times \mathbb{R}$,

$$|w(t,x) - m(x)| \leq C$$

for some globally Lipschitz continuous function m, and

$$\bar{A} \geq \max_{\alpha=L,R} \left(\min \overline{H}_\alpha \right).$$

Indeed: let w^ρ a corrector of (TCP) which is a solution, in particular, of

$$w_t^\rho + H(t,x,w_x^\rho) = \lambda_\rho \quad \text{in} \quad \mathbb{R}\times(-\rho,\rho)$$

and consider

$$\overline{w} = \limsup_{\rho\to+\infty}{}^*(w^\rho - w^\rho(0,0)), \quad \underline{w} = \liminf_{\rho\to+\infty}{}_*(w^\rho - w^\rho(0,0))$$

$$\text{and} \quad m = \lim_{\rho\to+\infty}(m^\rho - m^\rho(0)).$$

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$$\text{and} \quad m = \lim_{\rho\to+\infty}(m^\rho - m^\rho(0)).$$

Then

$$m - C \le \underline{w} \le \overline{w} \le m + C$$

and $\overline{w} - 2C$ and $\underline{w}$ are respectively sub and supersolution of

$$w_t + H(t,x,w_x) = \bar{A} \quad (t,x)\in\mathbb{R}\times\mathbb{R}.$$

A solution

$$\overline{w} - 2C \le w \le \underline{w}$$

is obtained by means of *Perron's method*.

(ii) Bound from below at infinity

If $\bar{A} > \max_{\alpha=L,R}\left(\min \bar{H}_\alpha\right)$, then there exists $\delta_0 > 0$ such that for every $\delta \in (0,\delta_0)$, there exists $\rho_\delta > \rho_0$ such that w satisfies

$$\begin{cases} w(t,x+h) - w(t,x) \geq (\bar{p}_R - \delta)h - C_\delta & \text{for } x \geq \rho_\delta, \quad h \geq 0, \\ w(t,x-h) - w(t,x) \geq (-\bar{p}_L - \delta)h - C_\delta & \text{for } x \leq -\rho_\delta, \quad h \geq 0, \end{cases}$$

where

$$\begin{cases} \bar{p}_R = \min E_R \\ \hat{p}_R = \max E_R \end{cases} \quad \text{with} \quad E_R := \left\{ p \in \mathbb{R}, \quad \overline{H}_R^+(p) = \overline{H}_R(p) = \bar{A} \right\}$$

$$\begin{cases} \bar{p}_L = \max E_L \\ \hat{p}_L = \min E_L \end{cases} \quad \text{with} \quad E_L := \left\{ p \in \mathbb{R}, \quad \overline{H}_L^-(p) = \overline{H}_L(p) = \bar{A} \right\}.$$

(iii) <u>Rescaling w</u>

For $\varepsilon > 0$, we set

$$w^{\varepsilon}(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x).$$

Then (along a subsequence $\varepsilon_n \to 0$) w^{ε} converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \le C\,|x-y| & \text{for all} \quad x, y \in \mathbb{R}, \\ \overline{H}_R(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_R \ge W_x \ge \bar{p}_R & \text{for } x \in (0,+\infty), \\ \overline{H}_L(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_L \le W_x \le \bar{p}_L & \text{for } x \in (-\infty, 0). \end{cases}$$

In particular, we have $W(0) = 0$ and

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \ge W(x) \ge \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}.$$

THEOREM [G. - Imbert - Monneau, Analysis & PDE (2015)]

Assume **(A0)**-**(A5)** and either **(B-i)** or **(B-ii)**. Assume that the initial datum u_0 is Lipschitz continuous and for $\varepsilon > 0$, let u^ε be the solution of (CP). Then u^ε converges locally uniformly to the unique flux-limited solution u^0 of

$$\begin{cases} u_t^0 + \overline{H}_L(u_x^0) = 0, & t > 0,\, x < 0, \\ u_t^0 + \overline{H}_R(u_x^0) = 0, & t > 0,\, x > 0, \\ u_t^0 + F_{\bar{A}}(u_x^0(t, 0^-), u_x^0(t, 0^+)) = 0, & t > 0,\, x = 0, \\ u^0(0, x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

It is sufficient to prove that

$$\begin{cases} \overline{u}(t,x) = \limsup_{\varepsilon \to 0} {}^*u^\varepsilon(t,x), \\ \underline{u}(t,x) = \liminf_{\varepsilon \to 0} {}_*u^\varepsilon(t,x) \end{cases}$$

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are respectively sub and supersolution.

Note that $\overline{u}$, $\underline{u}$ are well defined since

$$u_0(x) - Ct \leq u^\varepsilon(t,x) \leq u_0(x) + Ct$$

where $C = \sup_{\substack{(t,x)\in\mathbb{R}\times\mathbb{R} \\ |p|\leq L_0}} |H(t,x,p)|$ and L_0 is the Lipschitz constant of u_0.

The initial condition follows immediately.

Proof of convergence - subsolution case, $x = 0$

Let φ be a test function such that

$$(\overline{u}-\varphi)(t,x) < (\overline{u}-\varphi)(\bar{t},\overline{x}) = 0 \quad \forall (t,x) \in B_{\overline{r}}(\bar{t},\overline{x}) \setminus \left\{(\bar{t},\overline{x})\right\}$$

We argue by contradiction by assuming that

$$\varphi_t(\bar{t},\overline{x}) + \overline{H}\left(\overline{x},\varphi_x(\bar{t},\overline{x})\right) = \theta > 0,$$

where

$$\overline{H}\left(\overline{x},\varphi_x(\bar{t},\overline{x})\right) := \begin{cases} \overline{H}_R(\varphi_x(\bar{t},\overline{x})) & \text{if} \quad \overline{x} > 0, \\ \overline{H}_L(\varphi_x(\bar{t},\overline{x})) & \text{if} \quad \overline{x} < 0, \\ F_{\bar{A}}(\varphi_x(\bar{t},0^-),\varphi_x(\bar{t},0^+)) & \text{if} \quad \overline{x} = 0. \end{cases}$$

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We focus our attention to $\overline{x} = 0$. In this case the equations reads

$$\begin{aligned} \varphi_t(\bar{t},0) + \overline{H}\left(0, \varphi_x(\bar{t},0\right) &= \varphi_t(\bar{t},0) + F_{\bar{A}}(\varphi_x(\bar{t},0^-), \varphi_x(\bar{t},0^+)) \\ &= \varphi_t(\bar{t},0) + \max\left(\bar{A}, \overline{H}_L^+(\varphi_x(\bar{t},0^-)), \overline{H}_R^-(\varphi_x(\bar{t},0^+))\right) \\ &= \theta > 0. \end{aligned}$$

Key point (Imbert - Monneau '16)
Reduction to a single class of test function

To check the flux-limited junction condition, it is sufficient to consider very specific test functions:

$$\varphi(t,x) = \phi(t) + \bar{p}_L x 1_{\{x<0\}} + \bar{p}_R x 1_{\{x>0\}}$$

where ϕ is a C^1 function defined in $(0,+\infty)$. Hence

$$\begin{aligned}
\varphi_t(\bar{t},0) + F_{\bar{A}}(\varphi_x(\bar{t},0^-),\varphi_x(\bar{t},0^+)) &= \phi'(\bar{t}) + F_{\bar{A}}\left(\bar{p}_L,\bar{p}_R\right) \\
&= \phi'(\bar{t}) + \max\left(\bar{A}, \overline{H}_L^+(p_L), \overline{H}_R^-(p_R)\right) \\
&= \phi'(\bar{t}) + \bar{A} \\
&= \theta > 0.
\end{aligned}$$

Proof of convergence - subsolution case, $x = 0$

Let us consider a solution w of

$$w_t + H(t, x, w_x) = \bar{A}$$

and recall that

$$w^\varepsilon(t,x) = \varepsilon w\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}\right) \to W(x) \quad \textbf{V}\text{-shaped function.}$$

The *perturbed test function*

$$\varphi^\varepsilon(t,x) = \phi(t) + w^\varepsilon(t,x)$$

is a viscosity super-solution, for $r > 0$ small enough, of

$$\varphi^\varepsilon_t + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, \varphi^\varepsilon_x\right) = \frac{\theta}{2} \quad \text{in} \quad B_r(\bar{t}, 0).$$

Proof of convergence - subsolution case, $x = 0$

Fix $\kappa_r > 0$ and $\varepsilon > 0$ small enough so that

$$u^\varepsilon + \kappa_r \le \varphi^\varepsilon \quad \text{on} \quad \partial B_r(\bar{t}, 0).$$

By comparison principle

$$u^\varepsilon + \kappa_r \le \varphi^\varepsilon \quad \text{on} \quad B_r(\bar{t}, 0)$$

and passing to the limit as $(\varepsilon, t, x) \to (0, \bar{t}, 0)$ we get the following contradiction

$$\overline{u}(\bar{t}, 0) + \kappa_r \le \varphi(\bar{t}, 0) = \overline{u}(\bar{t}, 0). \quad \square$$

Remark

For the supersolution property we take

$$\varphi(t, x) = \phi(t) + \hat{p}_L x 1_{\{x<0\}} + \hat{p}_R x 1_{\{x>0\}}.$$

How the fraffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights?

Application to traffic light

How the fraffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights?

For N, $K \geq 1$ let:

$-\infty = b_0 < b_1 < b_2 < \cdots < b_N < b_{N+1} = +\infty$ junction points;

$0 = \tau_0 < \tau_1 < \cdots < \tau_K < 1 = \tau_{K+1}$ times;

$\ell_\alpha = b_{\alpha+1} - b_\alpha$ for $\alpha = 0, \ldots, N$.

How the fraffic flow on an ideal (infinite, straight) road is modified by the presence of a finite number of traffic lights?

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$\ell_\alpha = b_{\alpha+1} - b_\alpha$ for $\alpha = 0, \ldots, N$.

(C1) The Hamiltonian is given by

$$H(t,x,p) = \begin{cases} \overline{H}_\alpha(p) & \text{if } b_\alpha < x < b_{\alpha+1} \\ \max(\overline{H}^+_{\alpha-1}(p^-), \overline{H}^-_\alpha(p^+), a_\alpha(t)) & \text{if } x = b_\alpha, \alpha \neq 0. \end{cases}$$

(C2) The Hamiltonians $\overline{H}_\alpha$, for $\alpha = 0, \ldots, N$, are continuous, coercive and quasi-convex.

(C3) The flux limiters a_α, for $\alpha = 1, \ldots, N$ and $i = 0, \ldots, K$, satisfy

$$a_\alpha(s+1) = a_\alpha(s) \quad \text{with} \quad a_\alpha(s) = A^i_\alpha \quad \text{for all} \quad s \in [\tau_i, \tau_{i+1})$$

with $(A^i_\alpha)^{i=0,\ldots,K}_{\alpha=1,\ldots,N}$ satisfying $A^i_\alpha \geq \max_{\beta=\alpha-1,\alpha}\left(\min \overline{H}_\beta\right)$.

THEOREM [G. - Imbert - Monneau, Analysis & PDE (2015)]

Assume (C1)-(C3). Let u^ε be the solution of

$$\begin{cases} u_t^\varepsilon + H\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, u_x^\varepsilon\right) = 0 & \text{for } (t,x) \in (0,T)\times\mathbb{R} \\ u^\varepsilon(0,x) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases}$$

Then:

i) Homogenization: There exists some $\bar{A} \in \mathbb{R}$ such that u^ε converges locally uniformly as ε tends to zero towards the unique viscosity solution u^0 of

$$\begin{cases} u_t^0 + \overline{H}_L(u_x^0) = 0, & t>0,\ x<0, \\ u_t^0 + \overline{H}_R(u_x^0) = 0, & t>0,\ x>0, \\ u_t^0 + F_{\bar{A}}(u_x^0(t,0^-), u_x^0(t,0^+)) = 0, & t>0,\ x=0, \\ u^0(0,x) = u_0(x) & \text{for } x\in\mathbb{R}. \end{cases}$$

with

$$\overline{H}_L := \bar{H}_0, \quad \overline{H}_R := \bar{H}_N.$$

ii) Qualitative properties of $\bar{A}$: For $\alpha = 1, \dots, N$, $\langle a_\alpha \rangle$ denotes $\int_0^1 a_\alpha(s)\, ds$.
The effective limiter $\bar{A}$ satisfies the following properties.

- For all α, $\bar{A}$ is non-increasing w.r.t. ℓ_α.
- For $N = 1$,
$$\bar{A} = \langle a_1 \rangle.$$
- For $N \geq 1$,
$$\bar{A} \geq \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle.$$
- For $N \geq 2$, there exists a critical distance $d_0 \geq 0$ such that
$$\bar{A} = \max_{\alpha=1,\dots,N} \langle a_\alpha \rangle \qquad \text{if} \qquad \min_\alpha \ell_\alpha \geq d_0;$$
this distance d_0 only depends on $\max_{\alpha=1,\dots,N} \|a_\alpha\|_\infty$, $\max_{\alpha=1,\dots,N} \langle a_\alpha \rangle$ and the $\overline{H}_\alpha$.
- We have
$$\bar{A} \to \langle \bar{a} \rangle \quad \text{as} \quad (\ell_1, \dots, \ell_{N-1}) \to (0, \dots, 0)$$
where $\bar{a}(\tau) = \max_{\alpha=1,\dots,N} a_\alpha(\tau)$.

Thank you for your attention

For a function γ such that

$$\gamma \quad \text{is convex,} \quad \gamma \in C^2(\mathbb{R}) \quad \text{and} \quad \gamma' \geq \delta_0 > 0$$

we have

$$D^2_{pp}(\gamma \circ H_\alpha) > 0$$

if and only if

$$(\ln \gamma')'(\lambda) > -\frac{D^2_{pp} H_\alpha(x,p)}{(D_p H_\alpha(x,p))^2} \quad \text{for} \quad p = \pi^\pm_\alpha(x,\lambda) \quad \text{and} \quad \lambda \geq H_\alpha(x, p^0_\alpha)$$

where $\pi^\pm_\alpha(x,\lambda)$ are the partial inverse functions of H_α:

$$H_\alpha(x, \pi^\pm_\alpha(x,\lambda)) = \lambda \quad \text{such that} \quad \pm\, \pi^\pm_\alpha(x,\lambda) \geq 0.$$

(ii) Bound from below at infinity

If $\bar{A} > \max_{\alpha=L,R}\left(\min \bar{H}_\alpha\right)$, then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$, there exists $\rho_\delta > \rho_0$ such that w satisfies

$$\begin{cases} w(t,x+h) - w(t,x) \geq (\bar{p}_R - \delta)h - C_\delta & \text{for } x \geq \rho_\delta, \quad h \geq 0, \\ w(t,x-h) - w(t,x) \geq (-\bar{p}_L - \delta)h - C_\delta & \text{for } x \leq -\rho_\delta, \quad h \geq 0, \end{cases}$$

where

$$\begin{cases} \bar{p}_R = \min E_R \\ \hat{p}_R = \max E_R \end{cases} \quad \text{with} \quad E_R := \left\{p \in \mathbb{R}, \quad \overline{H}_R^+(p) = \overline{H}_R(p) = \bar{A}\right\}$$

$$\begin{cases} \bar{p}_L = \max E_L \\ \hat{p}_L = \min E_L \end{cases} \quad \text{with} \quad E_L := \left\{p \in \mathbb{R}, \quad \overline{H}_L^-(p) = \overline{H}_L(p) = \bar{A}\right\}.$$

We first control the *slopes* of correctors on truncated domains

$$w^\rho(t,x+h)-w^\rho(t,x)\geq(\bar p_R-\delta)h-C_\delta$$

and then $\rho\to+\infty$.

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$$w^\rho(t,x+h) - w^\rho(t,x) \geq (\bar{p}_R - \delta)h - C_\delta$$

and then $\rho \to +\infty$.
Let $\delta > 0$, then

$$|H(t,x,p) - H_R(t,x,p)| \leq \delta \quad \text{for} \quad x \geq \rho_\delta.$$

Since the map the map $p \mapsto \bar{H}_\alpha(p)$ is continuous and coercive, we can pick p_R^δ such that

$$\overline{H}_R(p_R^\delta) = \overline{H}_R^+(p_R^\delta) = \lambda_\rho - 2\delta$$

for $\rho \geq \rho_0$ and $\delta \leq \delta_0$, by choosing ρ_0 large enough and δ_0 small enough.

We first control the *slopes* of correctors on truncated domains

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for $\rho \geq \rho_0$ and $\delta \leq \delta_0$, by choosing ρ_0 large enough and δ_0 small enough.
We now fix $\rho \geq \rho_\delta$ and $x_0 \in [\rho_\delta, \rho]$. Take a $\mathbb{Z}^2$-periodic corrector v_R of

$$(v_R)_t + H_R(t,x,p_R^\delta + (v_R)_x) = \overline{H}_R(p_R^\delta), \quad (t,x) \in \mathbb{R} \times \mathbb{R}$$

so that $w_R = p_R^\delta x + v_R(t,x)$ solves

$$(w_R)_t + H_R(t,x,(w_R)_x) = \lambda_\rho - 2\delta, \quad (t,x) \in \mathbb{R} \times \mathbb{R}.$$

The restriction of w_R to $[\rho_\delta, \rho]$ satisfies

$$\begin{cases} (w_R)_t + H_R(t,x,(w_R)_x) \le \lambda_\rho - 2\delta & \text{for } (t,x) \in \mathbb{R} \times (\rho_\delta, \rho), \\ (w_R)_t + H_R^+(t,x,(w_R)_x) \le \lambda_\rho - 2\delta & \text{for } (t,x) \in \mathbb{R} \times \{\rho\}\,. \end{cases}$$

and

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The restriction of w_R to $[\rho_\delta, \rho]$ satisfies

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Now we remark that

$$v = w^\rho - w^\rho(0,x_0) \quad \text{and} \quad u = w_R - w_R(0,x_0) - 2C - 2\|v_R\|_\infty$$

satisfies

$$v(t,x_0) \ge -2C \ge u(t,x_0).$$

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Using a comparison principle for *mixed boundary value problem* we thus get for $x \in [x_0, \rho]$,

$$w^\rho(t,x) - w^\rho(t,x_0) \ge p_R^\delta(x - x_0) - C_\delta \ge (\bar{p}_R - \delta)h - C_\delta$$

where C_δ is a large constant which does not depend on ρ.

(iii) RESCALING w

For $\varepsilon > 0$, we set

$$w^\varepsilon(t,x) = \varepsilon w(\varepsilon^{-1}t, \varepsilon^{-1}x).$$

Then (along a subsequence $\varepsilon_n \to 0$) w^ε converges locally uniformly towards a function $W = W(x)$ which satisfies

$$\begin{cases} |W(x) - W(y)| \le C\,|x-y| & \text{for all} \quad x, y \in \mathbb{R}, \\ \overline{H}_R(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_R \ge W_x \ge \bar{p}_R & \text{for } x \in (0, +\infty), \\ \overline{H}_L(W_x) = \bar{A} \quad \text{and} \quad \hat{p}_L \le W_x \le \bar{p}_L & \text{for } x \in (-\infty, 0). \end{cases}$$

In particular, we have $W(0) = 0$ and

$$\hat{p}_R x 1_{\{x>0\}} + \hat{p}_L x 1_{\{x<0\}} \ge W(x) \ge \bar{p}_R x 1_{\{x>0\}} + \bar{p}_L x 1_{\{x<0\}}.$$

$$w^\varepsilon(t,x) = \varepsilon w\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) = \varepsilon m\left(\frac{x}{\varepsilon}\right) + \underbrace{\varepsilon\left[w\left(\frac{t}{\varepsilon},\frac{x}{\varepsilon}\right) - m\left(\frac{x}{\varepsilon}\right)\right]}_{O(\varepsilon)}$$

By *diagonal argument* can find a sequence $\varepsilon_n \to 0$ such that

$$w^{\varepsilon_n}(t,x) \to W(x) \quad \text{locally uniformly as } n \to +\infty,$$

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$$w^{\varepsilon_n}(t,x) \to W(x) \quad \text{locally uniformly as } n \to +\infty,$$

with $W(0) = 0$.
Moreover W satisfies

$$\overline{H}_R(W_x) = \bar{A} \ \text{ for } x > 0,$$
$$\overline{H}_L(W_x) = \bar{A} \ \text{ for } x < 0.$$

We deduce from bounds (ii) that in the case where $\bar{A} > \min \bar{H}_R$, for all $\delta > 0$ and $x > 0$

$$W_x \geq \bar{p}_R - \delta$$

and then

$$\bar{p}_R \leq W_x \leq \hat{p}_R. \tag{**}$$

In the case $\bar{A} = \min \bar{H}_R$, condition (**) is trivial.
Similarly, we can prove for $x < 0$ that

$$\hat{p}_L \leq W_x \leq \bar{p}_L.$$