Fully nonlinear uniformly elliptic/parabolic PDE with unbounded ingredients

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1. Introduction

Fully nonlinear elliptic/parabolic PDEs:

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega$$

$$u_t + F(x, t, u, Du, D^2u) = f(x, t) \quad \text{in } \Omega \times (0, T)$$

Elliptic PDE

$x \to F(x, r, \xi, X)$: measurable
$\forall$ fixed $r \in \mathbf{R}, \xi \in \mathbf{R}^n, X \in S^n$

$f \in L^p$
$\Downarrow$
L^p-viscosity solution
by
Caffarelli-Crandall-Kocan-Święch
(1996)
$\Uparrow$
Caffarelli (1989)
L^p- , Schauder regularity theory

Assumption (at least)

Elliptic $\quad \boldsymbol{p > \frac{n}{2}}$

Parabolic $\quad \boldsymbol{p > \frac{n+2}{2}}$

twice differentiable at a.e., continuity,

Notation

$$S^n := \{A : n \times n \text{ symmetric matirix}\}$$

$0 < \lambda \leq \Lambda$: unif. elliptic const.

Pucci extremal operators

$$\mathcal{P}^+(X) := \max\{-\mathrm{tr}(AX) \mid \lambda I \leq A \leq \Lambda I\}$$

$$\mathcal{P}^-(X) := \min\{-\mathrm{tr}(AX) \mid \lambda I \leq A \leq \Lambda I\}$$

$$F(x,u,Du,D^2u) = f(x) \quad \text{in } \Omega \qquad \text{(E)}$$

$$G(x,r,\xi,X) := F(x,r,\xi,X) - f(x)$$

Definition (parabolic case is similar)

$u \in C(\Omega)$: L^p-viscosity subsolution of (E)

$$\Leftrightarrow \begin{cases} \forall \phi \in W^{2,p}_{\mathrm{loc}}(\Omega), u - \phi : \text{maximum at } x \in \Omega \\ \qquad\qquad \Downarrow \\ ess \liminf_{y \to x} \left\{ G(y, u(y), D\phi(y), D^2\phi(y)) \right\} \leq 0 \end{cases}$$

$u \in C(\Omega)$: L^p-viscosity supersolution of (E)

$$\Leftrightarrow \begin{cases} \forall \phi \in W^{2,p}_{\mathrm{loc}}(\Omega), u - \phi : \text{minimum at } x \in \Omega \\ \Downarrow \\ ess \limsup_{y \to x} \left\{ G(y, u(y), D\phi(y), D^2\phi(y)) \right\} \geq 0 \end{cases}$$

Definition

u: L^p-strong subsolution (resp., supersolution) of (E)

$$\Leftrightarrow \begin{cases} u \in W^{2,p}_{\mathrm{loc}}(\Omega), \\ F(x, u(x), Du(x), D^2u(x)) \leq (\text{resp., } \geq)\ f(x) \quad \text{a.e. in } \Omega \end{cases}$$

Assumptions (only for Elliptic)

No dependence on u for F !

(A1) $F(x, 0, O) = 0$

(A2) $f \in L^p_+ \quad for\ p > \hat{p} \in [\frac{n}{2}, n)$: Escauriaza constant

(A3) uniformly elliptic

$$\mathcal{P}^-(X - Y) \leq F(x, \xi, X) - F(x, \xi, Y) \leq \mathcal{P}^+(X - Y)$$

(A4) $\exists \mu \in L^q_+$ such that $|F(x,\xi,O)| \le \mu(x)|\xi|^m$

$m = 1 \Leftrightarrow$ linear order
$m > 1 \Leftrightarrow$ superlinear order

u is L^p-viscosity subsolution of $F(x, Du, D^2u) = f$

$\Downarrow$

u is L^p-viscosity subsolution (resp., supersolution)

of

$$\mathcal{P}^-(D^2u) - \mu|Du|^m \le f^+ \quad \text{(resp., } \mathcal{P}^+(D^2u) + \mu|Du|^m \ge -f^-\text{)}$$

2. ABP maximum principle

Aleksandrov-Bakelman-Pucci maximum principle

$\Omega \subset \mathbf{R}^n$: bounded

$u \in C(\overline{\Omega})$: L^n-strong subsolution

of

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| \leq f(x) \quad \text{in } \Omega$$

$$\mu, f \in L^n_+(\Omega)$$

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+ + Ce^{C\|\mu\|_n^n}\|f\|_{L^n(\Omega[u])}$$

$\Omega[u]$: "upper contact set"

(Parabolic) ABP-Krylov-Tso maximum principle

$$\Pi := \Omega \times (0, T)$$

$u \in C(\overline{\Pi})$: L^{n+1}-strong subsolution
of

$$u_t + \mathcal{P}^-(D^2 u) - \mu(x,t)|Du| \le f(x,t) \quad \text{in } \Pi$$

$$\mu, f \in L_+^{n+1}(\Pi)$$

$$\sup_{\Pi} u \le \sup_{\partial_p \Pi} u^+ + C e^{C\|\mu\|_{n+1}^{n+1}} \|f\|_{L^{n+1}(\Pi[u])}$$

$\Pi[u]$: "parabolic upper contact set"

In what follows, we only consider the case only when

$$\boldsymbol{f} \in \boldsymbol{L^n}(\boldsymbol{\Omega}) \qquad \text{(elliptic)},$$

$$\boldsymbol{f} \in \boldsymbol{L^{n+1}}(\boldsymbol{\Pi}) \qquad \text{(parabolic)}.$$

ABP maximum principle for L^p-viscosity solutions

Elliptic

Caffarelli (1989), $f \in L^n \cap C(\Omega)$

Caffarelli-Crandall-Kocan-Święch (1996) $(m = 1)$
$\mu \in L^\infty$, $f \in L^n$

K-Święch(2007) $(m \geq 1)$
$\mu \in L^q$ $(q > n)$, $f \in L^n$

Remark ! when $m > 1$, the maximum principle fails in general.

If $\|\mu\|_q \|f\|_n^{m-1} \leq \delta_0$ for some $\delta_0 > 0$,
then
ABP holds when

$$q > n = p, \; m > 1$$

Application (Galise-K-Ley-Vitolo, 2016)

$$F(x, D^2u) + |Du|^m + |u|^{\alpha-1}u = f \quad \text{in } \mathbf{R}^n$$

$$\alpha > m > 1,\ f \in L^n_{loc}(\mathbf{R}^n)$$

$$\Downarrow$$

Uniquely existence of L^n-viscosity solution

$$\Uparrow$$

with a use of "Keller-Osserman" barriers

cf: Brezis (1984), $-\triangle u + |u|^{\alpha-1}u = f \quad \text{in } \mathbf{R}^n$

Parabolic ABP maximum principle

L. Wang (1992), $f \in L^{n+1} \cap C$

Crandall-Fok-Kocan-Święch (1998) $(m = 1)$
$\mu \in L^{\infty}$, $f \in L^{n+1}$

K-Święch (2007) $(m \geq 1)$
$\mu \in L^{q}$ $(q > n + 2)$, $f \in L^{n+1}$

Remark ! (parabolic) when $m > 1$ and $q = \infty$,
the smallness assumption is not necessary.

parabolic ABP holds when

(1) $q = \infty$, $p = n + 1$, $1 \le m < n + 2$
or
(2) $p = n + 1 < n + 2 < q < \infty$, $m = 1$

When $m > 1$ and $n + 2 < q < \infty$, the smallness is necessary;

If $\|f\|_{n+1}^{m-1} \|\mu\|_q \leq \exists \delta_0$, then ABP holds

when

$$p = n + 1 < n + 2 < q,\ 1 < m < (n + 2)(1 - \frac{n+1}{q})$$

Open question:

(1) Is there a counter-example for the maximum principle

parabolic case with $m > 1$?

($\mu \in L^q$ with $n + 2 < q < \infty$)

(2) $q = n$ (elliptic), $q = n + 1$ (parabolic) ??

3. (Weak) Harnack inequality

Notation: $Q_r := (-r/2, r/2)^n$

Elliptic (cf. 2009, K-Święch) $q > n = p$

$\exists C_0, \varepsilon_0, r > 0$ such that
for any $\mu \in L^q_+(Q_4)$ $f \in L^n_+(Q_4)$,
$u \in C(Q_4)$ is a nonnegative L^n-viscosity supersolution of

$$\mathcal{P}^+(D^2u) + \mu|Du| \geq -f \quad \text{in } Q_4,$$

$$\Downarrow$$

$$\|u\|_{L^r(Q_1)} \leq C_0 \left(\inf_{Q_1} u + \|f\|_{L^n(Q_4)} \right)$$

Recall Caffarelli's argument:

$$U(x) := \left(\inf_{Q_1} u + \varepsilon_0^{-1} \|f\|_{L^n(Q_4)}\right)^{-1} u(x)$$

1st reduction $\|f\|_n \le \varepsilon_0, \ \inf_{Q_1} u \le 1 \Rightarrow \|u\|_{L^r(Q_1)} \le C_0$

$$\Updownarrow$$

$$|\{x \in Q_1 \mid u(x) > M^k\}| \le \theta^k \ (\forall k \in \mathbf{N}) \quad (\exists M > 1, 0 < \exists \theta < 1)$$

$$\Uparrow$$

(discrete version of decay estimate for distribution function of u)

Construction of a key barrier function $\phi \in W^{2,n}(Q_4)$ and $\xi \in L^n(Q_4)$:

$$\begin{cases} \mathcal{P}^-(D^2\phi) - \mu|D\phi| = \xi & \text{in } Q_4, \\ \phi = 0 & \text{on } \partial Q_4, \\ \\ \phi \le -2 & \text{in } Q_3, \\ \xi = 0 & \text{in } Q_4 \setminus Q_1. \end{cases}$$

Let $\phi_0 \in W^{2,n}(Q_4)$ be a solution of

$$\begin{cases} \mathcal{P}^-(D^2\phi_0) - \mu|D\phi_0| = 0 & \text{in } Q_4 \setminus Q_{1/2}, \\ \phi_0 = 0 & \text{on } \partial Q_4, \\ \phi_0 = -1 & \text{on } \partial Q_{1/2}. \end{cases}$$

Classical strong maximum principle (Not very immediate)

$$\Downarrow$$

$\exists \rho > 0$ such that $\phi_0 \le -\rho$ in Q_3.

$$\Downarrow$$

Let ϕ be a smooth extension of $\frac{2}{\rho}\phi_0$ to $Q_{1/2}$.

$V := u + \phi$ is L^n-viscosity super solution
of

$$\mathcal{P}^+(D^2V) + \mu|DV| \geq f + \xi - \cancel{\mu|D\phi|}$$

$\Downarrow$ (ABP maximum principle)

$$-1 \geq \inf_{Q_4} V \geq -C(\|f\|_n + \|\xi\|_n + \cancel{\|\mu\|_n}) \quad \cancel{\|\mu\|_n << 1}$$

$$\|\xi\|_n \leq C|\{x \in Q_1 \mid u \leq \exists M_0\}|$$

$$\Rightarrow \quad 0 < \exists\theta \leq |\{x \in Q_1 \mid u(x) \leq M_0\}|$$

Parabolic (K-Święch-Tateyama, 201?) $(q > n + 2 > n + 1 = p)$

$$\Pi := Q_4 \times (0, 1)$$

$\exists C_0, r, \varepsilon_0 > 0$ such that

$$\begin{cases} f \in L_+^{n+1}(\Pi),\ \mu \in L_+^q(\Pi) \\ u \in C(\Pi) \text{ is a nonnegative } L^{n+1}\text{-viscosity supersolution} \\ \text{of } u_t + \mathcal{P}^+(D^2u) + \mu|Du| \geq -f \quad \text{in } \Pi \end{cases}$$

$$\Downarrow$$

$$\|u\|_{L^r(J_1)} \leq C_0 \left(\inf_{J_2} u + \|f\|_{L^{n+1}(Q_4 \times (0,1))} \right)$$

$$J_1 := Q_1 \times (0, 1/4), \quad J_2 := Q_1 \times (1/2, 1)$$

Reduction

$$\|f\|_{n+1} \le \varepsilon_0,\ \inf_{J_2} U \le 1 \Rightarrow \|u\|_{L^r(J_1)} \le C_0$$

$$\Updownarrow$$

Decay of distribution function

$$\Updownarrow$$

Discrete version of decay of distribution function

$$\Uparrow$$

Imbert-Silvestre (induction argument)

Construction of an L^{n+1}-strong solution ϕ and $\xi \in L^{n+2}(\Pi)$:

$$\begin{cases} \phi_t + \mathcal{P}^-(D^2\phi) - \mu|D\phi| = \xi & \text{in } \Pi := Q_4 \times (0,1), \\ \phi = 0 & \text{on } \partial_p \Pi, \\ \\ \phi \le -2 & \text{in } Q_3 \times (\frac{1}{4}, 1), \\ \xi = 0 & \text{in } Q_4 \times (0,1) \setminus Q_1 \times (0, \frac{1}{4}). \end{cases}$$

Let $\phi_0 \in W^{2,1}_{n+1}(Q_4 \times (0,1))$ be a solution of

$$\begin{cases} \frac{\partial}{\partial t}\phi_0 + \mathcal{P}^-(D^2\phi_0) - \mu|D\phi_0| = 0 & \text{in } Q_4 \times (0,1), \\ \phi_0 = 0 & \text{on } \partial Q_4 \times (0,1), \\ \phi_0 = \eta_0 & \text{on } Q_4 \times \{0\}, \end{cases}$$

where $\eta_0 \in C(Q_4)$ satisfies

$\eta_0 \leq 0$ in Q_4, $\eta_0 = 0$ in $Q_4 \setminus Q_{1/2}$, $\eta_0(0) = -1.$

Strong maximum principle (Not very immediate)

$\Downarrow$

$\exists \rho > 0$ such that $\phi_0 \leq -\rho$ in $Q_3 \times (\frac{1}{4}, 1)$.

Choose $\eta_1 \in C^\infty$ such that

$$\eta_1 = \begin{cases} 1 & \text{on } Q_4 \times (0,1) \setminus Q_1 \times (0, \frac{1}{4}), \\ 0 & \text{on } Q_{1/2} \times (0, \frac{1}{8}) \end{cases}$$

$$\Downarrow$$

Set $\phi := \frac{2}{\rho}\eta_1 \phi_0$: the desired function.

Corresponding $m > 1$ case, typical cases

Elliptic: $q > n = p$, $1 < m < 2 - \frac{n}{q}$, $\|\mu\|_n << 1$
(K-Święch 2009)

Parabolic: $q > n + 2 > n + 1 = p$, $1 < m < 2 - \frac{n+2}{q}$, $\|\mu\|_{n+1} << 1$
(K-Święch-Tateyama 201?)

$\Downarrow$

(Weak) Harnack inequality holds.

Remark

Elliptic case:

weak Harnack inequality $\Rightarrow$ Hölder continuity

Parabolic case:

weak Harnack inequality $\not\Rightarrow$ Hölder continuity ?

4. Applications

(1) strong maximum principle
(2) local maximum principle
(3) Harnack inequality
(4) Hölder continuity
(5) Boundary Harnack inequality

for

elliptic and parabolic,

$$\mathbf{1 < \boldsymbol{m} < 2 - \exists \boldsymbol{\tau}}$$

and more

Thank you for your attention.