Large Time Behavior of Periodic Viscosity Solutions of Integro-differential Equations

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Outline

Partial Integro Differential Equations (PIDEs)

Problem

Long Time Behavior of solutions of Partial Integro Differential Equations (PIDEs)

$$\begin{cases} u_t + F(x, Du, D^2u, \mathcal{I}[x,u]) = 0, & \text{in } \mathbb{R}^d \times (0,+\infty) \\ u(x,0) = u_0(x), & \text{in } \mathbb{R}^d \end{cases} \quad (1)$$

- The nonlinearity F is *degenerate elliptic*, i.e.

$$F(x,p,X,l_1) \leq F(x,p,Y,l_2) \text{ if } X \geq Y,\ l_1 \geq l_2, \quad \text{(E)}$$

- $\mathcal{I}[x,u]$ is an integro-differential operator of the form

$$\mathcal{I}[x,u] = \int_{\mathbb{R}^d} (u(x+z,t) - u(x,t) - Du(x,t)\cdot z1_B(z))\mu_x(dz)$$

$(\mu_x)_x$ family of Lévy measures s.t. $\sup_x \int_{\mathbb{R}^d} \min(1,|z|^2)\mu_x(dz) < \infty$.

Partial Integro Differential Equations (PIDEs)

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- $\mathcal{I}[x,u]$ is a *Lévy-Itô* operator of the form

$$\mathcal{J}[x,u] = \int_{\mathbb{R}^d} (u(x + j(x,z)) - u(x) - Du(x)\cdot j(x,z)1_B(z))\mu(dz)$$

with μ a Lévy measure and $j(x,z)$ the size of the jumps at x.

Partial Integro Differential Equations (PIDEs)

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$$F(x,p,X,l_1) \le F(x,p,Y,l_2) \text{ if } X \ge Y,\ \ l_1 \ge l_2, \quad \text{(E)}$$

- Fractional Laplacian of order $\beta \in (0,2)$

$$(-\Delta)^{\beta/2}u = \int_{\mathbb{R}^d} (u(x+z,t) - u(x,t) - Du(x,t)\cdot z1_B(z))\frac{dz}{|z|^{d+\beta}}$$

Partial Integro-Differential Equations. Lévy Processes

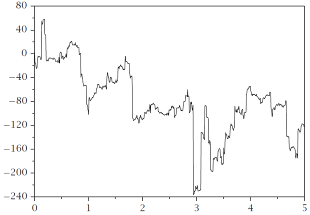

Figure: Stable Levy process. Jump discontinuities are represented by vertical lines.

The infinitezimal generator of a *Lévy process*

$$Lu(x) = \underbrace{b \cdot Du}_{drift} + \underbrace{\mathrm{tr}(AD^2u)}_{diffusion} + \underbrace{\int_{\mathbb{R}^d} \Big(u(x + j(x,z)) - u(x) - Du \cdot j(x,z) 1_B(z) \Big) \mu(dz)}_{jumps}.$$

Partial Integro-Differential Equations. Lévy Processes

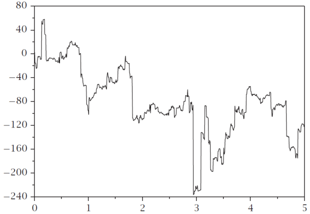

Figure: Stable Levy process. Jump discontinuities are represented by vertical lines.

The infinitezimal generator of a *wider class of Markov processes* of Courrège form

$$Lu(x) = \underbrace{c(x)u(x)}_{killing} + \underbrace{b(x)\cdot Du}_{drift} + \underbrace{\mathrm{tr}(A(x)D^2u)}_{diffusion} +$$
$$\underbrace{\int_{\mathbb{R}^d} \Big(u(x+z) - u(x) - Du\cdot z\, K(x,z) \Big) \mu_x(dz)}_{jumps}.$$

Partial Integro-Differential Equations. Lévy Processes

Example (Drift diffusion PIDEs)

$$u_t + (-\Delta)^{\beta/2} u + b(x) \cdot Du = f(x)$$

Example (Mixed PIDEs)

$$u_t - \Delta_{x_1} u + (-\Delta_{x_2})^{\beta/2} u + b(x) \cdot Du = f(x)$$

with $1 < \beta < 2$.

Example (Coercive PIDEs)

$$u_t - tr(A(x)D^2 u) - \mathcal{I}[u] + b(x)|Du|^m = f(x)$$

where

$$A(x) \geq a(x)I \geq 0,\ b(x) \geq b_0 > 0, m > 2.$$

LargeTime Behavior - periodic setting

Problem

Establish the long time behavior of periodic viscosity solutions:

$$u(x,t) = \lambda t + v(x) + o_t(1), \text{ as } t \to \infty.$$

- Nonlocal: Imbert, Monneau, Rouy '07
- Parabolic PDEs: Barles, Mitake, Ishii '09, Barles and Souganidis '06, '01, Barles Da Lio '05, Dirr and Souganidis, '95, Roquejoffre '01
- Hamilton Jacobi equations: Barles, Roquejoffre '06, Barles Souganidis '00 Lions, Papanicolau, Varadhan, Ishii' 10, Namah and Roquejoffre '99, Fathi '98, Fathi and Mather '00, Arisawa '97

LargeTime Behavior - periodic setting

Theorem (Barles, Chasseigne, C., Imbert '13)

Under suitable assumptions, the solution of the initial value problem

$$u_t + F(x, D^2 u, \mathcal{I}[u]) + H(x, Du) = f(x). \quad (2)$$

with $u_0 \in C^{0,\alpha}$ *and* $\mathbb{Z}^d$ *periodic satisfies*

$$u(x,t) - \lambda t \to v(x), \text{ as } t\infty \text{ uniformly in } x,$$

where v *is the unique periodic solution (up to addition of constants) of the stationary ergodic problem*

$$F(x, D^2 v, \mathcal{I}[v]) + H(x, Dv) = f(x) - \lambda \text{ in } \mathbb{R}^d. \quad (3)$$

The study relies in general on two main ingredients:

- Strong Maximum Principle
- Regularity of viscosity solutions

The Ergodic Problem

To solve the ergodic problem

$$\lambda + F(x, D^2 v, \mathcal{I}[v]) + H(x, Dv) = f(x) \text{ in } \mathbb{R}^d, \tag{4}$$

use the classical approximation

$$\delta v^\delta + F(x, D^2 v^\delta, \mathcal{I}[v^\delta]) + H(x, Dv^\delta) = f(x). \tag{5}$$

Perron's method and comparison principles: there exists a unique, bounded, periodic solution v^δ s.t. $||\delta v^\delta||_\infty \leq C$, hence $\delta v^\delta(0) \to \lambda$ as $\delta \to 0$.

Lemma

$$\tilde{v}^\delta(x) = v^\delta(x) - v^\delta(0)$$

is uniformly bounded and equicontinuous, hence $\tilde{v}^\delta \to v$ *(along subsequences).*

The Ergodic Problem - Compactness Mixed PIDEs

Proof.

Argue by contradiction: assume $||v^\delta||_\infty =: c_\delta \to \infty$ as $\delta \to 0$. Then the renormalized functions $w^\delta = \tilde{v}^\delta / c_\delta$ solve

$$\delta w^\delta + \frac{1}{c_\delta} F(x, c_\delta D^2 w^\delta, c_\delta \mathcal{I}[w^\delta]) + \frac{1}{c_\delta} H(x, c_\delta D w^\delta) = \frac{f(x) - \delta v_\delta(0)}{c_\delta}.$$

□

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Since $||w^\delta|| = 1$, solutions are uniformly $C^{0,\alpha}$ and up to a subsequence

$$w^\delta \to w, \text{ as } \delta \to 0, \text{ uniformly in x.}$$

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The limit w is $\mathbb{Z}^d$ periodic, $C^{0,\alpha}$, $w(0) = 0$, $||w||_\infty = 1$ and satisfies

$$F(x, D^2 w, \mathcal{I}[w]) + \overline{H}(x, Dw) = 0. \tag{6}$$

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Provided the limiting equation satisfies Strong Maximum Principle, $w \equiv 0$! □

The Ergodic Problem - Compactness Coercive PIDEs

Proof.

To fix ideas, let

$$H(x,p) = b(x)|p|^m \text{ for } m > 1.$$

The renormalized functions $w^\delta = \tilde{v}^\delta / c_\delta$ solve

$$\frac{1}{c_\delta^{m-1}} \delta w^\delta + \frac{1}{c_\delta^{m-1}} F(x, c_\delta D^2 w^\delta, c_\delta \mathcal{I}[w^\delta]) + b(x)|Dw^\delta|^m = \frac{f(x) - \delta v_\delta(0)}{c_\delta^m}.$$

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Solutions are uniformly $C^{0,\alpha}$ and up to a subsequence

$$w^\delta \to w, \text{ as } \delta \to 0, \text{ uniformly in x.}$$

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$$b(x)|Dw|^m = 0.$$

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The limit w is $\mathbb{Z}^d$ periodic, $C^{0,\alpha}$, $w(0) = 0$, $||w||_\infty = 1$ and satisfies

$$b(x)|Dw|^m = 0.$$

Provided $b(x) \geq b_0 > 0$ we get $Dw \equiv 0$, hence $w \equiv 0$! □

The Convergence

Proof.

Both $u(x,t)$ and $v(x)+\lambda t$ are solutions of (2). By comparison

$$m(t) := \sup_x (u(x,t) - \lambda t - v(x)) \searrow \bar{m}, \text{ as } \to \infty.$$

Take then the $\mathbb{Z}^d$ periodic functions $w(x,t) = u(x,t) - \lambda t$ and show $w(x, t+t_n) \to \bar{w}(x,t)$ as $t_n \to \infty$, where $\bar{w}$ solves

$$\begin{cases} \bar{w}_t + F(x, D^2\bar{w}, \mathcal{I}[\bar{w}]) + H(x, D\bar{w}) = f(x) - \lambda, & \text{in } \mathbb{R}^d \times (0, +\infty) \\ \bar{w}(x,0) = v(x), & \text{in } \mathbb{R}^d \end{cases} \tag{7}$$

Passing to the limit in $m(t+t_n)$ as $n \to \infty$ $\bar{m} = \sup_x(\bar{w}(x,t) - v(x))$ By the Strong Maximum Principle for the evolution equation above we get

$$\bar{w}(x,t) = v(x) + \bar{m}.$$

The conclusion follows. □

Strong Maximum Principle for PIDEs

Problem

Establish Strong Maximum Principle for Dirichlet boundary value problems

$$\begin{cases} u_t + F(x, t, Du, D^2u, \mathcal{J}[x, u]) = 0, \ \text{in } \Omega \times (0, T) \\ u = \varphi \text{ on } \Omega^c \times [0, T]. \end{cases} \tag{8}$$

Strong Maximum Principle and (Strong) Comparison Results

- nonlocal operators: Coville '08;
- elliptic second order equations: Bardi-Da Lio '01, '03, Da Lio '04, Nirenberg '53, Hopf '20s;

* comparison results and Jensen-Ishii's lemma: Jakobsen-Karlsen '06, Barles-Imbert '08, Jensen '88, Ishii '89, Crandall, Ishii, Lions '90s.

Strong Maximum Principle

Theorem (C. '11)

Anu $u \in USC(\mathbb{R}^d \times [0, T])$ viscosity subsolution of (8) that attains a maximum at $(x_0, t_0) \in \Omega \times (0, T)$ is constant in $\overline{\Omega} \times [0, t_0]$.

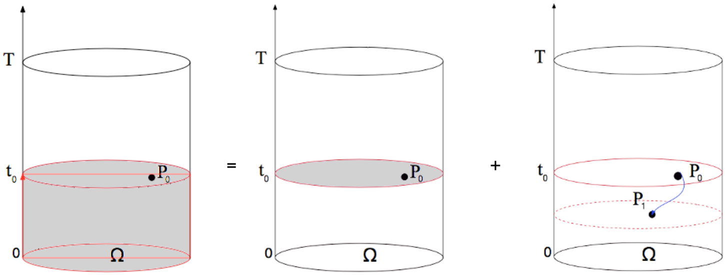


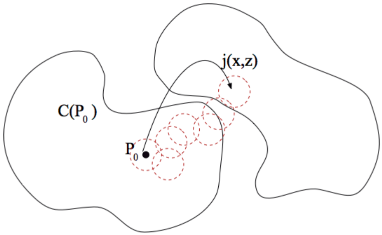
Figure: SMaxP = horizontal and vertical propagation of maxima.

Horizontal Propagation - Translations of Measure Supports

Theorem (C. '11)

If u attains a global maximum at $(x_0, t_0) \in \mathbb{R}^d \times (0, T)$*, then u is constant on* $\overline{\bigcup_{n\geq 0} A_n} \times \{t_0\}$ *with*

$$A_0 = \{x_0\}, \;\; A_{n+1} = \bigcup_{x \in A_n} (x + supp(\mu_x)). \tag{9}$$



$$\mu(dz) = \frac{dz}{|z|^{d+\beta}}.$$

Horizontal Propagation - Translations of Measure Supports

Example (Measures supported in the unit ball)

$$\mu(dz) = 1_B(z)\frac{dz}{|z|^{d+\beta}}.$$

Example (Measures charging two axis meeting at the origin)

$$\mu_x(dz) = 1_{\{z_1=\pm\alpha z_2\}}(z)\nu_x(dz),$$

Example (Pittfall of fractional diffusion on half space)

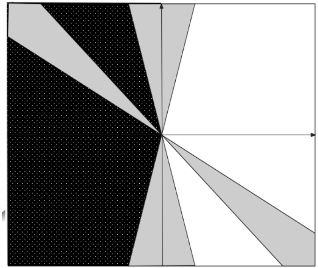
$$\mu(dz) = 1_{\{z_1\geq 0\}}(z)\frac{dz}{|z|^{d+\beta}}.$$

Horizontal Propagation - Nondegeneracy of the Measure

For any $x \in \Omega$ there exist $\beta \in (1,2), \eta \in (0,1)$ and a constant $C_\mu(\eta) > 0$ s.t. for $0 < |p| < R$

$$\int_{\mathcal{C}_{\eta,\gamma}(p)} |z|^2 \mu_x(dz) \geq C_\mu(\eta)\gamma^{\beta-2}, \forall \gamma \geq 1$$

$$\mathcal{C}_{\eta,\gamma}(p) = \{z;\ (1-\eta)|z||p| \leq |p \cdot z| \leq 1/\gamma\}$$



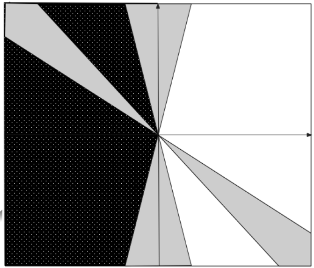
Theorem (C. '11)

Under suitable nondegeneracy and scaling assumptions, if a usc viscosity subsolution u attains a maximum at $P_0 = (x_0, t_0)$, then u is constant in the horizontal component of the domain, passing through point P_0.

Horizontal Propagation - Nondegeneracy of the Measure

Example (Overcome fractional diffusion on half space)

$$\mu(dz) = 1_{\{z_1 \geq 0\}}(z) \frac{dz}{|z|^{d+\beta}}, \beta > 1.$$



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Under suitable nondegeneracy and scaling assumptions, if a usc viscosity subsolution u attains a maximum at $P_0 = (x_0, t_0)$, then u is constant in the horizontal component of the domain, passing through point P_0.

Strong Maximum Principle

Example (SMaxP driven by differential terms)

$$u_t - \mathrm{tr}(\sigma(x)\sigma^*(x)D^2u) - c(x)\mathcal{I}[x,u] = f(x), \text{ in } \Omega \times (0,T)$$

where σ is a positive definite matrix and $c(x) \geq 0$.

Example (SMaxP driven by the nonlocal term)

$$u_t + b(x) \cdot Du + (-\Delta_x u)^{\beta/2} = f(x), \text{ in } \Omega \times (0,T)$$

where $b(\cdot)$ is a bounded vector field and the fractional exponent $\beta > 1$.

Example (SMaxP for mixed differential-nonlocal terms)

$$u_t - a_1(x)\Delta_{x_1} u + a_2(x)(-\Delta_{x_2} u)^{\beta/2} = f(x), \text{ in } \Omega \times (0,T)$$

where $a_1(x), a_2(x) \geq a_0 > 0$ and the fractional exponent $\beta > 1$.

Strong Comparison Principle

Theorem (C.)

If u and v are an usc viscosity subsolution, lsc viscosity supersolution s.t. $u - v$ attains a maximum at P_0, then $u - v$ is constant in $S(P_0)$.

Example (Mixed PIDEs)

$$u_t - a_1(x)\Delta_{x_1} u + a_2(x)(-\Delta_{x_2})^{\beta/2} u = f(x)$$

if the fractional exponent $\beta > 1$ and

$$a_1(x), a_2(x) \geq a_0 > 0.$$

Example (Coercive PIDEs)

$$u_t - \mathcal{I}[u] + b(x)|Du|^m = f(x)$$

if u is Lipschitz continuous, $b(x) \geq b_0 > 0$ and $m > 2$.

Hölder and Lipschitz Regularity

Problem

Viscosity solutions of PIDEs are $C^{0,\alpha}$/Lipschitz continuous in space (dep. on β)

$$||u||_{C^{0,\alpha}} \leq C||u||_\infty.$$

Main approaches for proving the Hölder regularity of *viscosity solutions* of local/nonlocal equations

- *Harnack estimates*, for uniformly elliptic equations: Guillen Schwab '11, Silvestre Imbert '14, Silvestre Cardaliaguet '12, Silvestre Vicol '12, Silvestre '06,'11, Caffarelli, Silvestre '09, '11 Caffarelli Cabré '95
  Regularity and ABP estimates for a larger class of PIDEs left open!
- *direct viscosity methods* such as Ishii-Lions's '90, for degenerate elliptic equations: Barles, Chasseigne and Imbert '11, Cardaliaguet-Rainer '10
  Lipschitz or further regularity, e.g. $C^{1,\alpha}$ left open!

Model equations for Hölder and Lipschitz regularity

Advection fractional diffusion

$$u_t + (-\Delta u)^{\beta/2} + b(x) \cdot Du = f$$

- Subcritical case $\beta > 1$: for $b \in L^\infty$ the solution is Lipschitz continuous.
- Critical case $\beta = 1$: for $b \in C^\tau$, $\tau > 0$ the solution is C^β.
- Supercritical case $\beta < 1$: for $b \in C^{1-\beta+\tau}$, the solution is C^β.

Model equations for Hölder and Lipschitz regularity

Fractional difussion with superlinear gradient growth

$$u_t - tr(A(x)D^2u) + a_2(x)(-\Delta)^{\beta/2}u + b(x)|Du|^k = f$$

with nondegenerate diffusion $A(x) \geq a_1(x)I$, with

$$a_1(x) + a_2(x) \geq a_0 > 0.$$

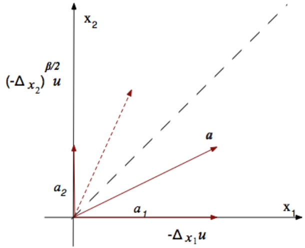
- When $\beta > 1$: for $b \in C^{0,\tau}$ and $k \leq \tau + \beta$ the solution is Lipschitz.
- When $\beta > 1$: for $b \in L^\infty$ and $k \leq \beta$ the solution is Lipschitz continuous.
- When $\beta < 1$: for $b \in C^{1-\beta+\tau}$, and $k \leq \beta$ the solution is C^β.

Horizontal Propagation - Nondegeneracy of the Measure

Mixed ellipticity

$$-a_1(x_1)\Delta_{x_1} u + a_2(x_2)(-\Delta_{x_2} u)^{\beta/2} = f(x_1, x_2)$$

with $a_i(x) \geq a_0 > 0$.



Problem

Solutions are Lipschitz continuous in the x_1-variable and Hölder, resp. Lipschitz continuous in the x_2 variable if $\beta \leq 1$, resp. $\beta > 1$.

Partial Regularity

We give both *Hölder and Lipschitz regularity results* of viscosity solutions for a *general class of mixed integro-differential equations* of the type

$$-a_1(x_1)\Delta_{x_1} u \quad -a_2(x_2)\mathcal{I}_{x_2}[x,u] - \mathcal{I}[x,u]$$
$$+b_1(x_1)|D_{x_1} u_1|^{k_1} + b_2(x_2)|D_{x_2} u|^{k_2} + |Du|^n + cu = f(x).$$

Theorem (Barles, Chasseigne, C., Imbert '12)

Any periodic continuous viscosity solution u

(a) *is Lipschitz in the x_2 variable, if $\beta > 1$ and $k_2 \leq \beta$, $k_1 = 1$, $n \geq 0$;*

(b) *is $C^{0,\alpha}$ with $\alpha < \frac{\beta - k_2}{1 - k_2}$, if $\beta \leq 1$ and $k_2 < \beta$, $k_1 = 1$, $n \geq 0$.*

(c) *If $b_2 \in C^{0,\tau}(\mathbb{R}^{d_2})$, then we can deal wtih growth up to $k_2 \leq \beta + \tau$.*

The Lipschitz/Hölder constant depends on $||u||_\infty$, on the dimension d, the constants associated to the Lévy measures and on the functions a_2, b_2 and f.

Insight - Proof of the Regularity Results

Classical argument for Hölder continuity: show that

$$\max_{x,y} (u(x) - u(y) - \phi(|x-y|)) < 0.$$

where for Hölder regularity the control function is given by

$$\phi(|x-y|) = L|x-y|^{\alpha}$$

whereas for Lipschitz by

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For partial regularity, use classical regularity arguments in one set of variables, and uniqueness type arguments in the other variables:

$$\psi_\varepsilon(x,y) = u(x_1,x_2) - u(y_1,y_2) - \phi(x_1-y_1)$$

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$$\psi_{\varepsilon}(x,y) = u(x_1,x_2) - u(y_1,y_2) - \phi(x_1-y_1) - \frac{|x_2-y_2|^2}{\varepsilon^2}$$

Global regularity

A priori estimates. The regularity results can be extended to superlinear cases, by a gradient cut-off argument.

$$\begin{aligned} -a_1(x_1)\Delta_{x_1} u \quad & - a_2(x_2)\mathcal{I}_{x_2}[x,u] - \mathcal{I}[x,u] \\ +b_1(x_1)|D_{x_1} u_1| \quad & + b_2(x_2)|D_{x_2} u| \quad + |Du|^n + cu = f(x). \end{aligned}$$

Theorem (Barles, Chasseigne, C., Imbert '12)

Any periodic continuous viscosity solution u

(a) *is Lipschitz continuous, if* $\beta > 1$ *and* $k_2 = 1,\ k_1 = 1,\ n \geq 0$*;*

(b) *is* $C^{0,\alpha}$ *continuous with* $\alpha < \frac{\beta_2 - k_2}{1 - k_2}$, *if* $\beta \leq 1$ *and* $k_2 = 1,\ k_1 = 1,\ n \geq 0$.

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$$+b_1(x_1)|D_{x_1} u_1|^{k_1} + b_2(x_2)|D_{x_2} u|^{k_2} + |Du|^n + cu = f(x).$$

Theorem (Barles, Chasseigne, C., Imbert '12)

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(a) *is Lipschitz continuous, if* $\beta > 1$ *and* $k_2 \le \beta_i$, $k_1 \le 2$, $n \ge 0$;

(b) *is* $C^{0,\alpha}$ *continuous with* $\alpha < \frac{\beta_2 - k_2}{1 - k_2}$, *if* $\beta \le 1$ *and* $k_2 < \beta_i$, $k_1 \le 2$, $n \ge 0$.

Extensions of the regularity results

- the non-periodic setting
- parabolic case

$$u_t + F_0(\dots, \mathcal{I}[x,u]) + F_1(\dots, \mathcal{I}_{x_1}[x,u]) + F_2(\dots, \mathcal{J}_{x_2}[x,u]) = f(x)$$

- fully nonlinear Bellman - Isaacs equations

$$\sup_{\gamma\in\Gamma}\inf_{\delta\in\Delta}\Big(F_0^{\gamma,\delta}(\dots,\mathcal{J}^{\gamma,\delta}) + F_1^{\gamma,\delta}(\dots,\mathcal{J}_{x_1}^{\gamma,\delta}) + F_2^{\gamma,\delta}(\dots,\mathcal{J}_{x_2}^{\gamma,\delta}) - f^{\gamma,\delta}(x)\Big) = 0$$

- multiple nonlinearities.

Thank you for your attention!