Laboratoire de Mathématiques et
Physique Théorique (UMR CNRS 7350)
Fédération Denis Poisson

Has HJNET brought something for the understanding of Hamilton-Jacobi Equations with discontinuities ?

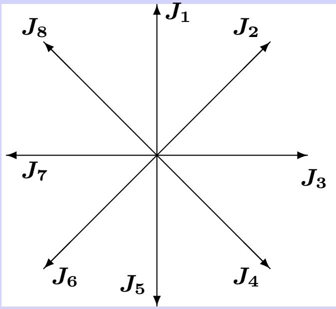
G. Barles

Based on "joint" works with

A. Briani, E. Chasseigne & C. Imbert

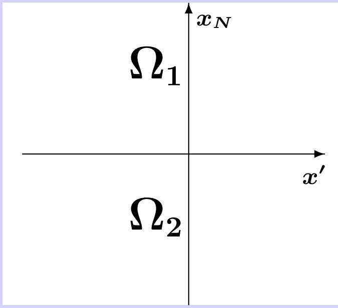
In the HJNET project, two different (?) types of Hamilton-Jacobi-Bellman problems

1. Networks :



Achdou, Imbert, Monneau, Oudet, Tchou, Nguyen (Vinh), Zidani in HJnet, with co-authors (Camilli, Cutri, Schieborn & Marchi)

2. Multi-domains :



Briani, Chasseigne, Forcadel, Rao, Tchou, Zidani and co-authors (Hermosilla, Siconolfi)

The simplest common framework : two half-spaces and an infinite horizon control problem

x_N

$$\Omega_1 := \{x \in \mathbb{R}^N \ : \ x_N > 0\}$$

Dynamic : b_1

Running cost : l_1

x'

$$\Omega_2 := \{x \in \mathbb{R}^N \ : \ x_N < 0\}$$

Dynamic : b_2

Running cost : l_2

Same constant discount factor in Ω_1 and Ω_2 : λ

What is clear and not clear ?

x_N

$\Omega_1 := \{x \in \mathbb{R}^N \ : \ x_N > 0\}$ $\qquad$ $H_1 = 0$ in Ω_1

What is the correct condition on $\mathcal{H} := \{x \in \mathbb{R}^N \ : \ x_N = 0\}$?

How to define the dynamic/cost on the interface ?

x'

$\Omega_2 := \{x \in \mathbb{R}^N \ : \ x_N < 0\}$ $\qquad$ $H_2 = 0$ in Ω_2

with

$$H_i(x, u, p) = \sup_{\alpha_i} \{-b_i(x, \alpha_i) \cdot p + \lambda u - \ell_i(x, \alpha_i)\}$$

(Strange) answer :

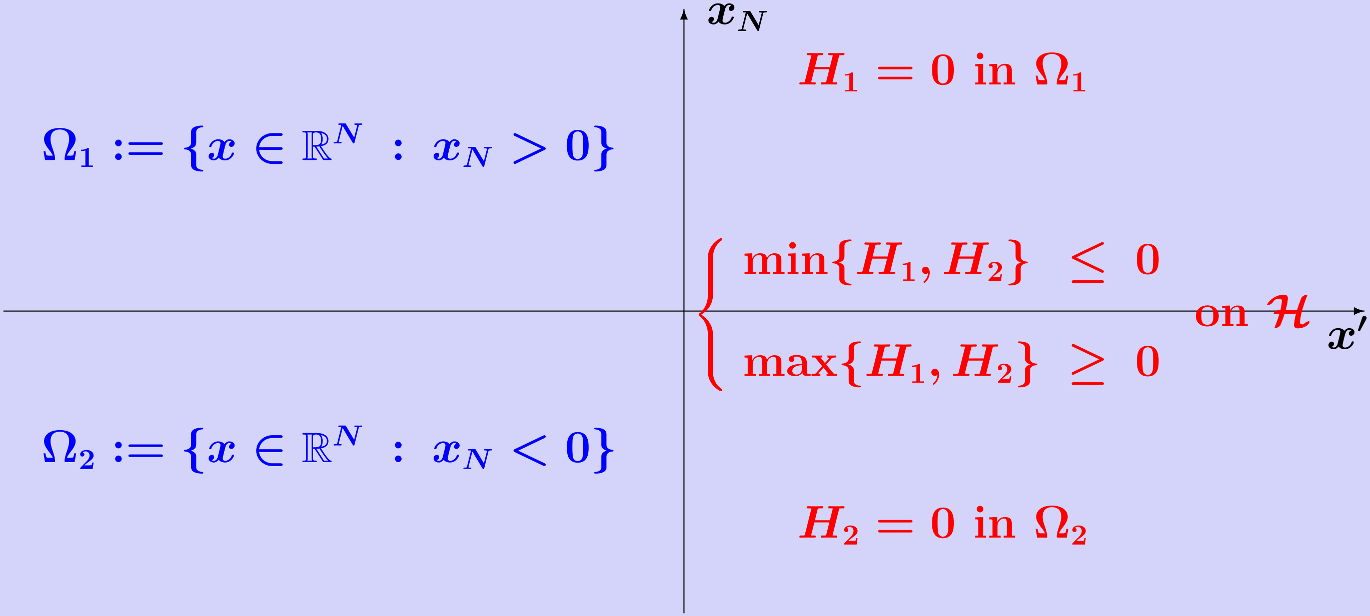
(i) It depends how you see the picture : as $\mathbb{R}^N$ with a discontinuity somewhere or as two half spaces with no special links

(ii) Mathematically, do you choose

- **test-function as in $\mathbb{R}^N$ (C^1-test-functions in $\mathbb{R}^N$) ?**

- **or test-functions with are piecewise C^1 ? (C^1 in $\overline{\Omega_1}$ and $\overline{\Omega_2}$ and continuous at the interface)**

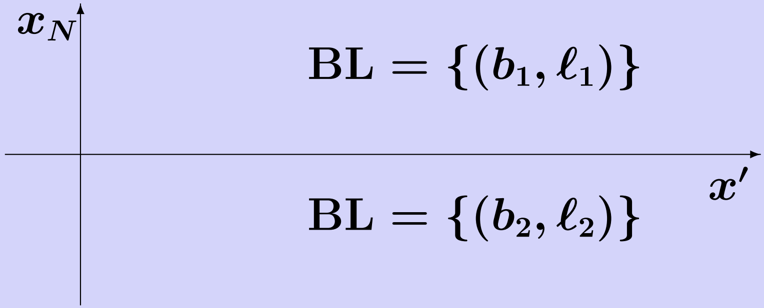
The most classical approach : C^1-test-functions in $\mathbb{R}^N$. Ishii's formulation



Questions : Is "the" value function the unique solution of this problem ? If not, can we identify the minimal and maximal solutions ? Are additional conditions required on $\mathcal{H}$ to obtain uniqueness ?

But how to define the control problem(s) ?

First Step : Differential Inclusion



To satisfy the right assumptions, one has to extend BL on $\{x_N = 0\}$ by upper-semi-continuity and convexity.

On $\{x_N = 0\}$, the dynamic and cost are given by

$$(\mu b_1 + (1-\mu) b_2, \mu \ell_1 + (1-\mu)\ell_2) \ .$$

And the associated trajectories which play a role [staying on $\{x_N = 0\}$] are those for which

$$(\mu b_1 + (1-\mu) b_2) \cdot e_N = 0$$

Control Problems

Dynamic-Cost

$$(\dot{X}_{x_0}(t), \dot{L}_{x_0}(t)) \in \mathrm{BL}(X_{x_0}(t)), \quad X_{x_0}(0) = x_0 \in \mathbb{R}^N$$

Value Functions

$$\mathrm{U}^-(x_0) := \inf_{\text{all traj.}} \left(\int_0^{+\infty} \dot{L}_{x_0}(t) e^{-\lambda t} dt \right)$$

$$\mathrm{U}^+(x_0) := \inf_{\text{reg. traj.}} \left(\int_0^{+\infty} \dot{L}_{x_0}(t) e^{-\lambda t} dt \right)$$

where "regular trajectories" are whose which uses tangent vector fields $(\mu b_1 + (1-\mu) b_2) \cdot e_N = 0$

But ONLY with $b_1 \cdot e_N \leq 0,\ b_2 \cdot e_N \geq 0$

THEOREM (Briani-Chasseigne-B) :
Under controlability assumptions

(i) U^+ is the maximal solution of the HJB Equation,

(ii) U^- is the minimal solution of the HJB Equation.

(iii) Moreover, if we add the subsolution condition

$$H_T(x, u, Du) \leq 0 \quad \text{on } \{x_N = 0\}$$

where

$$H_T(x, u, p) = \sup_{\alpha}\{-b \cdot p + \lambda u - \ell\}$$

with $b = \mu b_1 + (1 - \mu)b_2$ and $b \cdot e_N = 0$, then U^- is the unique solution of this new problem.

Conclusions

– We have a complete description of the structure of solutions of the HJB Equation

– No uniqueness in general for the HJB Equation (in the sense of Ishii)

– Uniqueness (and even comparison) if we add an extra condition on the discontinuity set ($\{x_N = 0\}$)

Problem : the uniqueness/comparison proof are half-pde, half-control...

The Network approach : Imbert-Monneau

x_N

$H_1 = 0$ in Ω_1

$\Omega_1 := \{x \in \mathbb{R}^N \ : \ x_N > 0\}$

$$\begin{cases} \max\{H_1^+, H_2^-\} \ \leq \ 0 \\ \max\{H_1^+, H_2^-\} \ \geq \ 0 \end{cases} \text{on } \mathcal{H}$$

x'

$\Omega_2 := \{x \in \mathbb{R}^N \ : \ x_N < 0\}$

$H_2 = 0$ in Ω_2

where in H_1^+, H_2^-, you keep only the "inward" dynamics to Ω_1, Ω_2 respectively.

NB : Test-function is ϕ_1 on Ω_1, ϕ_2 on Ω_2 and H_1, H_1^+ use ϕ_1, H_2, H_2^- use ϕ_2.

THEOREM (Imbert-Monneau) : Under (not so useful) controlability assumptions, there exists a unique solution of this problem which is the value function U^N given by

$$\inf \int_0^t \left\{ l_1(X(t),\alpha_1(t))\mathbb{1}_{\mathbb{I}_1}(t) + l_2(X(t),\alpha_2(t))\mathbb{1}_{\mathbb{I}_2}(t) \right\} e^{-\lambda t} dt$$

with $\mathbb{I}_i$ is the set when $X(t) \in \overline{\Omega_i}$ and b_i is used (if $X(t) \in \mathcal{H}$).

$$\dot{X}(t) := b_1(X(t),\alpha_1(t))\mathbb{1}_{\mathbb{I}_1}(t) + b_2(X(t),\alpha_2(t))\mathbb{1}_{\mathbb{I}_2}(t)$$

In other words, you use only the dynamic/cost (b_1, l_1) or (b_2, l_2). No mixing on $\mathcal{H}$!

Second idea of Imbert-Monneau : Flux limiter !

In terms of control : you add a specific control problem on $\mathcal{H}$.

In term of pdes, you change the junction condition in

$$\begin{cases} \max\{G, H_1^+, H_2^-\} \leq 0 \\ \max\{G, H_1^+, H_2^-\} \geq 0 \end{cases} \quad \text{on } \mathcal{H} \, .$$

THEOREM (Imbert-Monneau+Nguyen (Vinh) revisited by Briani-Chasseigne-Imbert-B) :

(i) You have a comparison result and therefore a unique solution U^G for this problem.

(ii) $U^- \leq U^+ \leq U^N$ in $\mathbb{R}^N$ (without G).

(iii) $U^- = U^G$ in $\mathbb{R}^N$ if $G = H_T$.

(iv) $U^+ = U^G$ in $\mathbb{R}^N$ if $G = H_T^{reg}$.

NB : complete pde proofs !

Useful ? (because my co-authors had doubts at the beginning...)

Corollary : The solutions obtained through the vanishing viscosity method converge to U^+.

This was a puzzling open problem...

Proof : For getting the right condition on $\mathcal{H}$, do it only for the (non-smooth) test-function of the comparison proof ! (an old idea of PLL...)

Last remark : the comparison result can be extended to second-order problems, provided that there is no diffusion anymore on $\mathcal{H}$...

= first-order junction conditions !

Some answers to key questions ?

Can a "universal pde approach" exist ? NO

(i) the conditions on the junction depend on the type of problem you have (for example, whether you minimize or maximize in the control problem),

(ii) the junction conditions are also connected to the problem you want to solve via the G,

(iii) the pde proofs use "some kind of convexity".

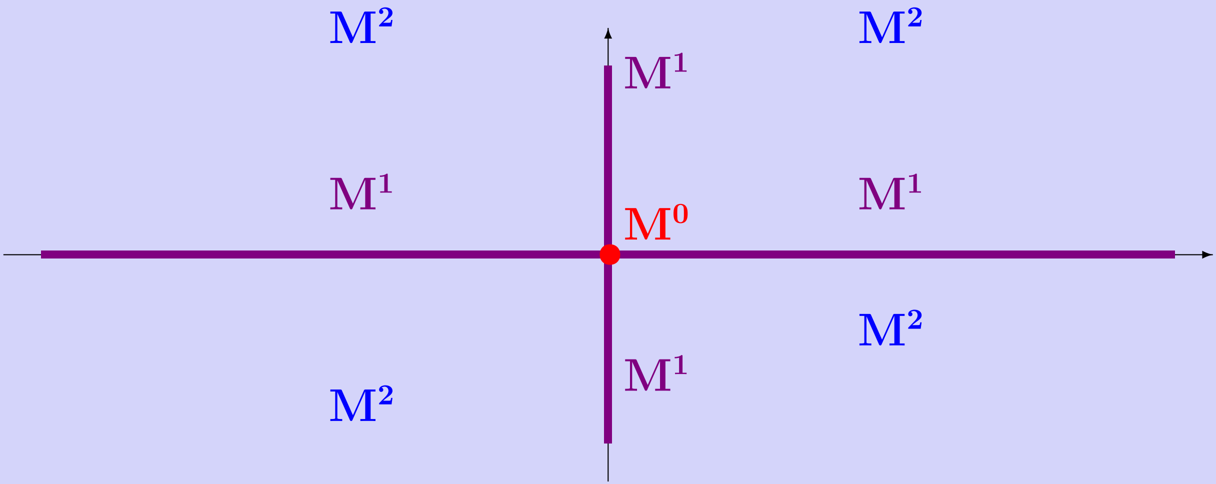
Is our "half-control – half-pde proof" completely useless ? NO (or not yet ?)
The network/pde proof is giving more information in this case but no yet in the case of stratified problems, = problems with some higher co-dimensions type discontinuities.

Stratified domains : Bressan-Hong revisited by Chasseigne-B

We write $\mathbb{R}^N$ as a partition

$$\mathbb{R}^N = \mathrm{M}^0 \cup \mathrm{M}^1 \cup \cdots \cup \mathrm{M}^N ,$$

where, for any k, M^k is a k-dimensional submanifold of $\mathbb{R}^N$ with suitable assumptions.



With a suitable formulation of the problem and "natural" assumptions, we have

(i) A comparison result and therefore a unique solution for the stratified problem (value function of the associated control problem).

(ii) A (completely new) stability result.

What are the "natural" assumptions ?

–Normal controlability : in a neighborhood of each M^k, one can reach M^k. Key assumption in any result : (a) to prove that the value function is solution, (b) to prove stability, (c) to prove comparison.

–Tangential regularity : each Hamiltonian should satisfy classical assumptions for comparison on M^k.

But the proof is half-pde, half control...

Now you can (perhaps) answer in a better way to the original question

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