Long time behavior of Mean Field Games

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The MFG system

Given a finite horizon $T > 0$, we consider the MFG system

$$(MFG) \qquad \left\{ \begin{array}{ll} -\partial_t u^T - \sigma \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0,T) \times \mathbb{T}^d \\ \partial_t m^T - \sigma \Delta m^T - \operatorname{div}(m^T D_p H(x, Du^T)) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ u^T(T,x) = g(x, m^T(T)), \ m^T(0,\cdot) = m_0 & \text{in } \mathbb{T}^d \end{array} \right.$$

where

- $u = u(t,x)$ and $m = m(t,x)$ are the unknown,
- $\sigma \geq 0$ is the viscosity,
- $H = H(x,p) : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is a smooth, unif. convex in p, Hamiltonian,
- $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are “smooth",
  ($\mathcal{P}(\mathbb{T}^d) =$ the set of Borel probability measures on $\mathbb{T}^d$)
- $m_0 \in \mathcal{P}(\mathbb{T}^d)$ is a smooth positive density

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The limit problem

Let (u^T, m^T) be the solution to

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Study the limit as $T \to +\infty$ of the pair (u^T, m^T).

- **Motivation :** one numerically observes that the pair (u^T, m^T) quickly stabilizes as T is large (when f and g are monotone).
- In contrast with similar problem for HJ equation, **initial** and **terminal** conditions for the system.

The limit problem

One expects that (u^T, m^T) “converges" to the solution of the ergodic MFG problem

$$(MFG-erg) \qquad \left\{ \begin{array}{ll} \bar\lambda - \sigma\Delta \bar u + H(x,D\bar u) = f(x,\bar m) & \text{in } \mathbb{T}^d \\ -\sigma\Delta \bar m - \mathrm{div}(\bar m D_pH(x,D\bar u)) = 0 & \text{in } \mathbb{T}^d \\ \bar m \geq 0 \quad \text{in } \mathbb{T}^d, \qquad \int_{\mathbb{T}^d} \bar m = 1 & \end{array} \right.$$

where now the unknown are $\bar\lambda$, $\bar u = \bar u(x)$ and $\bar m = \bar m(x)$.

- For the Fokker-Plank equation driven by a vector-field V :

$$\partial_t m^T - \sigma\Delta m^T - \mathrm{div}(m^T V(x)) = 0 \qquad \text{in } (0,T)\times\mathbb{T}^d$$

(exponential) convergence of m^T to the ergodic measure is well-known.

- For HJ equations :

$$\partial_t u - \sigma\Delta u + H(x, Du) = f(x) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d$$

  - Ergodic constant $\bar\lambda$, convergence of $u(T,\cdot)/T$: Lions-Papanicolau-Varadhan, ...
  - Convergence of $u(T,\cdot) - \bar\lambda T$: Fathi, Roquejoffre, Fathi-Siconolfi, Barles-Souganidis, ...

- For the MFG system, convergence of u^T/T and m^T :
  - Lions (Cours in Collège de France)
  - Gomes-Mohr-Souza (discrete setting)
  - C.-Lasry-Lions-Porretta (viscous setting), C. (Hamilton-Jacobi)
  - Turnpike property (Samuelson, Porretta-Zuazua, Trélat,...)

- Long-time behavior not known so far.

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Outline

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1. The long time average
2. The master equation
3. The long-time behavior

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Standing assumptions

- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, with :

$$C^{-1} I_d \leq D^2_{pp} H(x,p) \leq C I_d \text{ and } D_x H(x,p) \cdot p \geq -C|p|^2 \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- the maps $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (f(x,m) - f(x,m'))d(m-m')(x) \geq 0, \; \int_{\mathbb{T}^d} (g(x,m) - g(x,m'))d(m-m')(x) \geq 0$$

- the maps f, g are Lipschitz continuous and $f(\cdot, m), g(\cdot, m)$ are uniformly bounded in $C^{2+\alpha}$ for some $\alpha \in (0,1)$.

Example. If f is of the form :

$$f(x,m) = \int_{\mathbb{R}^d} \Phi(z,(\rho \star m)(z))\rho(x-z)dz,$$

where

- $\star$ denotes the usual convolution product (in $\mathbb{R}^d$),
- $\Phi = \Phi(x,r) : \mathbb{T}^d \times [0,+\infty) \to \mathbb{R}$ is a smooth map, nondecreasing w.r. to r,
- ρ is a smooth, even function with compact support.

Then f satisfies our conditions.

Indeed, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (f(x,m) - f(x,m'))d(m-m')(x)$$
$$= \int_{\mathbb{T}^d} \left[\Phi(y,\rho\star m(y)) - \Phi(y,\rho\star m'(y))\right] \left(\rho\star m(y) - \rho\star m'(y)\right) dy \geq 0,$$

since ρ is even and Φ is nondecreasing with respect to the second variable. So f is monotone.

Basic convergence result

Let U^T, M^T be the scaled functions on $\mathbb{T}^d \times [0,1]$:

$$U^T(t,x) := u^T(tT,x) \qquad ; \qquad M^T(t,x) := m^T(tT,x)$$

Theorem (C.-Lasry-Lions-Porretta, '13)

As $T \to +\infty$,

- $\frac{U^T(t,\cdot)}{T}$ converges uniformly to $-(1-t)\bar{\lambda}$ in $\mathbb{T}^d \times [0,1]$.
- $M^T(t)$ converges to $\bar{m}$ in $L^2(\mathbb{T}^d)$, uniformly on compact intervals of $(0,1)$.

Actually, convergence with an exponential rate.

Ingredients of proof

Lemma

The map u^T is uniformly semi-concave in space.

In particular, u^T is uniformly Lipschitz continuous in space.

Proof : Comes from the regularity of $f(\cdot, m^T)$ and the fact that $H = H(x, p)$ is uniformly convex in p.

Main energy inequality. Computing $\frac{d}{t}\int_{\mathbb{T}^d} u^T(t,x)m^T(t,x)dx$ and integrating we get the main energy inequality :

$$\int_0^T\int_{\mathbb{T}^d} C^{-1}(m^T+\bar{m})|Du^T-D\bar{u}|^2+(f(x,m^T)-f(x,\bar{m}))(m^T-\bar{m})\;dxdt$$
$$\leq -\left[\int_{\mathbb{T}^d}(u^T-\bar{u})(m^T-\bar{m})dx\right]_0^T$$

where the RHS is bounded because

- at $t=T$, $\left|\int_{\mathbb{T}^d}(u^T(T,x)-\bar{u}(x))(m^T(T,x)-\bar{m}(x))dx\right| \leq 2(\|g\|_\infty+\|\bar{u}\|_\infty)$
- at $t=0$, $\left|\int_{\mathbb{T}^d}(u^T(0,x)-\bar{u}(x))(m^T(0,x)-\bar{m}(x))dx\right|$

$$\leq \left|\int_{\mathbb{T}^d}(u^T(0,x)-\int_{\mathbb{T}^d}u^T(0))(m^T(0,x)-\bar{m}(x))dx\right|+2\|\bar{m}\|_\infty$$

$$\leq C\mathrm{Lip}(u^T(0,\cdot))+2\|\bar{m}\|_\infty$$

So

$$\int_0^T \int_{\mathbb{T}^d} \frac{(m^T + \bar{m})}{2} |Du^T - D\bar{u}|^2 + (f(x, m^T) - f(x, \bar{m}))(m^T - \bar{m}) \, dxdt \leq C$$

where $\bar{m} > 0$.

In particular

$$\lim_{T \to +\infty} \frac{1}{T} \int_0^T \int_{\mathbb{T}^d} |Du^T - D\bar{u}|^2 \, dxdt = 0 \; ,$$

which is equivalent to

$$\lim_{T \to +\infty} \int_0^1 \int_{\mathbb{T}^d} |DU^T - D\bar{u}|^2 \, dxdt = 0 \; ,$$

This implies

- the convergence of m^T,
- and then the convergence of u^T/T.

Outline

The master equation

In order to study the long time behavior of the solution (u^T, m^T), we use the master equation, which subsumes the MFG system into a single equation :

$$\left\{\begin{array}{l} -\partial_t U - \Delta_x U + H(x, D_x U) - f(x,m) \\ \qquad - \int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U\right] \, dm(y) + \int_{\mathbb{T}^d} D_m U \cdot D_p H(y, D_x U) \, dm(y) = 0 \\ \qquad\qquad \text{in } [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \\ U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array}\right.$$

where $U = U(t,x,m) : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$.

Derivatives in the space of measures

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed for the Monge-Kantorovich distance

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi(y)\, d(m - m')(y),$$

where the supremum is taken over all Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$ with a Lipschitz constant bounded by 1.

Given $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$, we consider 2 notions of derivatives :

- The directional derivative $\frac{\delta U}{\delta m}(m, y)$ (see, e.g., Mischler-Mouhot)
- The intrinsic derivative $D_m U(m, y)$ (see, e.g., Otto, Ambrosio-Gigli-Savaré, Lions)

Directional derivative

A map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$U(m') - U(m) = \int_0^1 \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}((1-s)m + sm', y)d(m'-m)(y)ds.$$

Note that $\frac{\delta U}{\delta m}$ is defined up to an additive constant. We adopt the normalization convention

$$\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m, y)dm(y) = 0.$$

Intrinsic derivative

If $\dfrac{\delta U}{\delta m}$ is of class $\mathcal{C}^1$ with respect to the second variable, the intrinsic derivative $D_m U : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}^d$ is defined by

$$D_m U(m, y) := D_y \frac{\delta U}{\delta m}(m, y)$$

For instance, if $U(m) = \int_{\mathbb{T}^d} g(x)dm(x)$, then $\frac{\delta U}{\delta m}(m, y) = g(y) - \int_{\mathbb{T}^d} gdm$ while $D_m U(m, y) = Dg(y)$.

Remarks.

- The directional derivative is fruitful for computations.
- The intrinsic derivative encodes the variation of the map in $\mathcal{P}(\mathbb{T}^d)$. For instance :

$$\|D_m U\|_\infty = Lip\ U$$

Strengthened assumptions

- $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is smooth, with :

$$C^{-1} I_d \leq D^2_{pp} H(x,p) \leq C I_d \text{ and } D_x H(x,p)\cdot p \geq -C|p|^2 \qquad \text{for } (x,p) \in \mathbb{T}^d \times \mathbb{R}^d.$$

- the maps $f, g : \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ are monotone : for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} (f(x,m) - f(x,m'))d(m-m')(x) \geq 0, \int_{\mathbb{T}^d} (g(x,m) - g(x,m'))d(m-m')(x) \geq 0$$

- the maps f, g are $\mathcal{C}^1$: there exists $\alpha \in (0,1)$ such that

$$\sup_{m \in \mathcal{P}(\mathbb{T}^d)} \left(\|f(\cdot,m)\|_{2+\alpha} + \left\| \frac{\delta f(\cdot,m,\cdot)}{\delta m} \right\|_{(2+\alpha,2+\alpha)} \right) + \mathrm{Lip}(\frac{\delta f}{\delta m}) \ < \ \infty.$$

  and the same for g.
- The derivatives of the maps f, g are symmetric :

$$\frac{\delta f}{\delta m}(x,m,y) = \frac{\delta f}{\delta m}(y,m,x) \text{ and } \frac{\delta g}{\delta m}(x,m,y) = \frac{\delta g}{\delta m}(y,m,x)$$

The master equation

It is the backward equation

$$(\mathbf{M}) \qquad \left\{ \begin{array}{l} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H(x, D_x U(t,x,m)) - f(x,m) \\ \qquad - \displaystyle\int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U\right](t,x,m,y)\, dm(y) \\ \qquad + \displaystyle\int_{\mathbb{T}^d} D_m U(t,x,m,y) \cdot D_p H(y, D_x U(t,y,m))\, dm(y) \;=\; 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } (t,x,m) \in [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(T,x,m) = G(x,m), \qquad \text{for } (x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{array} \right.$$

Theorem (C.-Delarue-Lasry-Lions, 2015)

Under our assumptions, the master equation (**M**) has a unique classical solution.

In fact, holds for the master equation with common noise.

Advantage of the master equation

- The solution is smooth, defined for all times and subsumes the MFG system.

- As noticed by Lions, the MFG system forms the characteristics of the master equation : given $(t_0, m_0) \in \mathcal{P}(\mathbb{T}^d)$, let $(m(t))$ solve the McKean-Vlasov equation :

$$\left\{ \begin{array}{ll} \partial_t m - \Delta m - \mathrm{div}\,(m D_p H(x, DU(t,x,m))) = 0 & \text{in } (t_0, T) \times \mathbb{T}^d \\ m(t_0) = m_0 \quad \text{in } \mathbb{T}^d \end{array} \right.$$

and set $u(t,x) = U(t,x,m(t))$.

Then (u, m) is the solution of the MFG system starting from m_0 at time t_0.

- Conversely, given an initial position $(t_0, m_0) \in (0, T) \times \mathcal{P}(\mathbb{T}^d)$ and (u, m) a solution of the MFG system starting from m_0 at time t_0, the map U defined by

$$U(t_0, x, m_0) := u(t_0, x)$$

solve the master equation.

Outline

The master cell-problem

Theorem (C.-Porretta, 2016)

There is a unique constant λ such that the master cell-problem :

$$\lambda - \Delta_x \chi(x,m) + H(x, D_x\chi(x,m)) - \int_{\mathbb{T}^d} \mathrm{div}(D_m\chi(x,m))dm + \int_{\mathbb{T}^d} D_m\chi(x,m).H_p(x,,D_x\chi(x,m))dm = f(x,m) \qquad \text{in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$$

has a (weak) solution.

Moreover, if $(\bar\lambda, \bar u, \bar m)$ is the solution to the ergodic MFG system :

$$\left\{\begin{array}{ll} \bar\lambda - \Delta \bar u + H(x, D\bar u) = f(x,\bar m) & \text{in } \mathbb{T}^d \\ -\Delta \bar m - \mathrm{div}(\bar m D_p H(x, D\bar u)) = 0 & \text{in } \mathbb{T}^d \\ \bar m \geq 0 \quad \text{in } \mathbb{T}^d, \qquad \int_{\mathbb{T}^d} \bar m = 1 & \end{array}\right.$$

then

$$\bar\lambda = \lambda \qquad \text{and} \qquad D_x\chi(x,\bar m) = D\bar u(x) \qquad \forall x \in \mathbb{T}^d.$$

The construction of χ relies as usual on a discounted equation :

$$\delta U^\delta - \Delta_x U^\delta + H(x, D_x U^\delta) - \int_{\mathbb{T}^d} \operatorname{div}(D_m U^\delta) dm + \int_{\mathbb{T}^d} D_m U^\delta . H_p(x, D_x U^\delta(x,m)) dm = f(x,m) \text{ in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d).$$

The main step is the proof of the Lipschitz estimate :

Proposition

There is a constant C, depending on the data only, such that

$$\left\| D_m U^\delta(\cdot, m, \cdot) \right\|_{2+\alpha, 1+\alpha} \leq C.$$

In particular, $U^\delta(\cdot,\cdot)$ is uniformly Lipschitz continuous.

Difficulty : equation for U^δ neither coercive nor elliptic in m.

Idea of proof

Representation formulas. Fix $m_0 \in \mathcal{P}(\mathbb{T}^d)$ a initial condition and (u^δ, m^δ) the associated solution of the discounted MFG system :

$$\left\{\begin{array}{l} -\partial_t u^\delta + \delta u^\delta - \Delta u^\delta + H(x, Du^\delta) = f(x, m^\delta(t)) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t m^\delta - \Delta m^\delta - \mathrm{div}(m^\delta H_p(x, Du^\delta)) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ m^\delta(0,\cdot) = m_0 \text{ in } \times\mathbb{T}^d, \qquad u^\delta \text{ bounded.} \end{array}\right.$$

For any smooth map μ_0 with $\int_{\mathbb{T}^d} m_0 = 0$, one can show that

$$\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(x, m_0, y)\mu_0(y)dy = w(0,x),$$

where (w, μ) is the unique solution to the linearized system

$$\left\{\begin{array}{l} -\partial_t w + \delta w - \Delta w + H_p(x, Du^\delta).Dw = \dfrac{\delta f}{\delta m}(x, m^\delta(t))(\mu(t)) \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \partial_t \mu - \Delta\mu - \mathrm{div}(\mu H_p(x, Du^\delta)) - \mathrm{div}(m^\delta H_{pp}(x, Du^\delta)Dw) = 0 \qquad \text{in } (0,+\infty)\times\mathbb{T}^d \\ \mu(0,\cdot) = \mu_0 \text{ in } \mathbb{T}^d, \qquad w \text{ bounded.} \end{array}\right.$$

The standard energy estimate gives

$$\int_0^{\infty} e^{-\delta t} m^{\delta} |Dw|^2 \leq C \int_{\mathbb{T}^d} w(0)\mu_0.$$

Estimate of μ. Recall that μ solves

$$\begin{cases} \partial_t \mu - \Delta\mu - \mathrm{div}(\mu H_p(x, Du^{\delta})) - \mathrm{div}(m^{\delta} H_{pp}(x, Du^{\delta}) Dw) = 0 & \text{in } (0, +\infty) \times \mathbb{T}^d \\ \mu(0, \cdot) = \mu_0 \text{ in } \mathbb{T}^d. \end{cases}$$

Following [CLLP], there exists $\omega > 0$, depending on $\|D_p H\|_\infty$ only, such that, for $t \geq 0$,

$$\|\mu(t)\|_1 \leq C \left(\|\mu_0\|_{(C^{2+\alpha})'} e^{-\omega t} + \left(\int_0^t \int_{\mathbb{T}^d} \left| m^{\delta} D^2_{pp} H(x, Du^{\delta}) Dw \right|^2 \right)^{\frac{1}{2}} \right).$$

Using the energy estimate then yields to

$$\|\mu(t)\|_1 \leq C \left(\|\mu_0\|_{(C^{2+\alpha})'} e^{-\omega t} + e^{\delta t/2} \left(\int_{\mathbb{T}^d} w(0)\mu_0 \right)^{\frac{1}{2}} \right).$$

Estimate of w. Next we apply estimates in [CLLP] to $w = w(t,x)$, which satisfies

$$-\partial_t w + \delta w - \Delta w + H_p(x, Du^\delta).Dw = \int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(x, m^\delta(t), y)\mu(t,y)dy \qquad \text{in } (0,+\infty)\times\mathbb{T}^d.$$

We obtain :

$$\|w(t) - \langle w(t)\rangle\|_\infty \leq Ce^{\delta t}\int_t^\infty e^{-\omega(s-t)-\delta s}\,\|\mu(s)\|_{(C^{2+\alpha})'}\,ds.$$

Using the estimate for μ, we get therefore :

$$\|w(t) - \langle w(t)\rangle\|_\infty \leq Ce^{\delta t}\left(\|\mu(0)\|_{(C^{2+\alpha})'} + \left(\int_{\mathbb{T}^d} w(0)\mu(0)\right)^{\frac{1}{2}}\right).$$

Key step : estimate of $\langle w(t)\rangle$. We first claim that $\xi(t) := \int_{\mathbb{T}^d} w(t,x)m^\delta(t,x)dx = 0$ for $t \geq 0$. Indeed

$$\frac{d}{dt}\xi(t) + \delta\xi(t) = \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(x, m^\delta(t), y)\mu(t,y)m^\delta(t,x)dydx,$$

where, as $\frac{\delta f}{\delta m}(\cdot, m, \cdot)$ is symmetric, the RHS equals

$$\int_{\mathbb{T}^d}\left(\int_{\mathbb{T}^d} \frac{\delta f}{\delta m}(y, m^\delta(t), x)m^\delta(t,x)dx\right)\mu(t,y)dy = 0$$

by convention on the derivative. This easily yields the claim :

$$\int_{\mathbb{T}^d} w(t,x)m^\delta(t,x)dx = 0 \qquad \forall t \geq 0.$$

Conclusion. Using the previous estimates we find :

$$\|w(0)\|_{C^{2+\alpha}} \leq C\left(\|\mu_0\|_{(C^{2+\alpha})'} + \left(\int_{\mathbb{T}^d} w(0)\mu_0\right)^{\frac{1}{2}}\right)$$

Whence

$$\|w(0)\|_{C^{2+\alpha}} = \left\|\int_{\mathbb{T}^d} \frac{\delta U^\delta}{\delta m}(\cdot, m_0, y)\mu_0(y)dy\right\|_{C^{2+\alpha}} \leq C\|\mu_0\|_{(C^{2+\alpha})'}.$$

Main convergence results

Let χ be a solution to the master cell problem. We consider the solution to the backward master equation

$$\begin{cases} -\partial_t U(t,x,m) - \Delta_x U(t,x,m) + H(x, D_x U(t,x,m)) - \int_{\mathbb{T}^d} \mathrm{div}(D_m U(t,x,m)) dm \\ \qquad + \int_{\mathbb{T}^d} D_m U(t,x,m).H_p(t,x,m) dm = f(x,m) \qquad \text{in } (-\infty, 0) \times \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d), \\ U(0,x,m) = G(x,m) \text{ in } \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d) \end{cases}$$

Theorem (C.-Porretta, 2016)

There exists a constant $c \in \mathbb{R}$ such that

$$\lim_{t \to -\infty} U(t,x,m) + \bar{\lambda} t = \chi(x) + c,$$

uniformly with respect to $(x,m) \in \mathbb{T}^d \times \mathcal{P}(\mathbb{T}^d)$.

The Theorem implies in particular the convergence of the solution to the MFG system as $T \to +\infty$.

Corollary

Let c be the constant given in the Theorem. For $T > 0$ and $m_0 \in \mathcal{P}(\mathbb{T}^d)$, let (u^T, m^T) be the solution to the MFG system :

$$\left\{ \begin{array}{ll} -\partial_t u^T - \sigma \Delta u^T + H(x, Du^T) = f(x, m^T(t)) & \text{in } (0,T) \times \mathbb{T}^d \\ \partial_t m^T - \sigma \Delta m^T - \mathrm{div}(m^T D_p H(x, Du^T)) = 0 & \text{in } (0,T) \times \mathbb{T}^d \\ u^T(T, x) = g(x, m^T(T)), \; m^T(0, \cdot) = m_0 & \text{in } \mathbb{T}^d \end{array} \right.$$

Then

$$\lim_{T \to +\infty} u^T(0, x) + \bar{\lambda} T = \chi(x) + c$$

Proof. Indeed, $u^T(0, x) = U(-T, x, m_0)$. The result follows immediately.

Conclusion

We have established the long time behavior of the solution of the MFG system and of the master equation.

Open problems :

- Analysis in more realistic setting (with boundary conditions, non constant diffusion matrices,...)
- Convergence of the discounted MFG system and of the discounted master equation.
- Convergence without the symmetry condition or in the non-monotone setting.

Thank you !