Controlling Structured Population Models

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Controlling Structured Population Models

Introduction

Measure Valued Conservation Laws [P.Gwiazda & M.Rosińska]

- Well Posedness
- Control

Structured Populations on Graphs [M.Garavello]

- Well Posedness
- Control

Introduction: Structured Population Models

u population density

Introduction: Structured Population Models

$u = u(t)$	population density
t	time

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	age

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	age

$$\partial_t u + \partial_x u = 0$$

Introduction: Structured Population Models

$u = u(t, x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left([\text{growth/aging}]\, u\right) = 0$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t, x, \textstyle\int u)\, u \right) = 0$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t, x, \textstyle\int u)\, u\right) = [\text{death}]$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u\right) = -d(t,x,u,\textstyle\int u)$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u \right) = -d(t,x,u,\textstyle\int u)$$
$$u(t,0) = [\text{birth}]$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t,x,\int u)\, u \right) = -d(t,x,u,\int u)$$
$$u(t,0) = b(t,\int u)$$

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$$\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u \right) = -d(t,x,u,\textstyle\int u)$$

$$u(t,0) = b(t,\textstyle\int u)$$

$$u(0,x) = \text{[initial datum]}$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\partial_t u + \partial_x \left(g(t,x,\int u)\, u\right) = -d(t,x,u,\int u)$$
$$u(t,0) = b(t,\int u)$$
$$u(0,x) = \bar{u}(x)$$

Introduction: Structured Population Models

$u = u(t,x)$	population density
t	time
x	biological age/size/(trait)

$$\begin{aligned} &\partial_t u + \partial_x \left(g(t,x,\textstyle\int u)\, u\right) = -d(t,x,u,\textstyle\int u) \\ &u(t,0) = b(t,\textstyle\int u) \\ &u(0,x) = \bar{u}(x) \end{aligned}$$

Balance Law with Boundary
NonLocal flow
NonLocal source
NonLocal boundary data

Introduction: Conservation Laws - Analytic Theory

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

x	$\in$	$\mathbb{R}^N$	space
t	$\in$	$\mathbb{R}^+$	time
u	$\in$	$\mathbb{R}^n$	unknown
f	is	smooth	flow ($n \times N$)
g	is	smooth	o.d.e.

Introduction: Conservation Laws - Analytic Theories

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Scalar MultiD

$n = 1$ and $N \geq 1$

Systems in 1D

$n \geq 1$ and $N = 1$

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$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

Systems in 1D

$n \geq 1$ and $N = 1$

$$\partial_t u + \partial_x f(u) = 0$$

Introduction: Conservation Laws - Analytic Theories

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

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Scalar MultiD

$n = 1$ and $N \geq 1$

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

Existence

(Kružkov: Mat.Sb., 1970)

Systems in 1D

$n \geq 1$ and $N = 1$

$$\partial_t u + \partial_x f(u) = 0$$

Existence

(Glimm: CPAM, 1965)

Introduction: Conservation Laws - Analytic Theories

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

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Scalar MultiD

$n = 1$ and $N \geq 1$

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

Existence

(Kružkov: Mat.Sb., 1970)

Dependence on data

(Kružkov: Mat.Sb., 1970)

Systems in 1D

$n \geq 1$ and $N = 1$

$$\partial_t u + \partial_x f(u) = 0$$

Existence

(Glimm: CPAM, 1965)

Dependence on data

(Bressan & c.: 1995, 2000)

Introduction: Conservation Laws - Analytic Theories

$$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$$

x	$\in$	$\mathbb{R}^N$	space
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Scalar MultiD	Systems in 1D
$n = 1$ and $N \geq 1$	$n \geq 1$ and $N = 1$
$\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)$	$\partial_t u + \partial_x f(u) = 0$
Existence	Existence
(Kružkov: Mat.Sb., 1970)	(Glimm: CPAM, 1965)
Dependence on data	Dependence on data
(Kružkov: Mat.Sb., 1970)	(Bressan & c.: 1995, 2000)
Dependence on f, g	Dependence on f
(Colombo, Mercier & Rosini: CMS, 2009)	(Bianchini, Colombo: PAMS, 2002)

Introduction: Conservation Laws – Keywords

$$\boxed{\partial_t u + \mathrm{div}_x f(t,x,u) = g(t,x,u)}$$

$$x \in \mathbb{R}^N \qquad u \in \mathbb{R}^n$$

$$\boxed{N = 1,\ n \geq 1} \qquad \boxed{N \geq 1,\ n = 1}$$

x regularity: $\mathbf{L^1}$, $\mathbf{L^\infty}$, $\mathbf{BV}$

Norm: $\|\cdot\|_{\mathbf{L^1}}$

t regularity: $\mathbf{L^1}$ – Lipschitz

parameters regularity: $\mathbf{L^1}$ – Lipschitz

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
    - (Chen & Christoforou: PAMS, 2007)
    - (Christoforou: JHDE, 2007)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
    - (Amadori & Shen: Comm.PDE, 2009)
    - (Guerra & Shen: JDE, 2014)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
    - (Colombo, Corli & Rosini: ZAMM, 2007)
    - (Colombo, Herty & Mercier: COCV, 2011)
    - (Dong & Tong: NHM, 2011)
    - (Blandin & Goatin: Numer.Math., 2015)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
    - (Colombo, Garavello & Mercier: M3AS, 2011)
    - (Colombo & Mercier: Acta Math.Sc., 2011)
    - (Piccoli & Tosin: ARMA, 2011)
    - (Hoogendoorn et al.: Physica A, 2014)
    - (Goatin & Rossi: Preprint, 2015)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
    - (Betancourt, Burger, Karlsen & Tory: Nonlin., 2011)
    - (Amorim, Colombo & Teixeira: M2AN, 2015)
    - (Aggarwal, Colombo & Goatin: SINUM, 2015)

Introduction: NonLocal Conservation Laws

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
    - (Coclite & Coclite: JDE, 2015)
    - (Coclite & Karlsen: J.Func.An., 2015)
    - (Coclite, Ridder & Risebro: Preprint, 2016)

Introduction: Structured Population

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
  - Measure Valued Conservation Laws
    - (Carrillo, Colombo, Gwiazda & Ulikowska: JDE, 2012)
    - (Gwiazda, Jamróz & Marciniak-Czochra: SIMA, 2012)
    - (Canizo, Carrillo & Cuadrado: Acta Appl.Math., 2013)
    - (Piccoli & Rossi: ARMA, 2014)

Introduction:

- NonLocal Conservation Laws:
  - Elastodynamics
  - Granular Materials
  - Vehicular Traffic
  - Crowd Dynamics
  - Numerical Methods
  - Degasperis–Procesi, Camassa–Holm & Ostrovsky–Hunter
  - Measure Valued Conservation Laws

- Structured Population
  - 1150+ items in MathSciNet (100 books)
    - (Webb: Theory of Nonlinear Age-Dependent Population Dynamics, 1985)
    - (Diekmann & Heesterbeek: Mathematical Epidemiology of Infectious Diseases, 2000)
    - (Perthame: Transport Equations in Biology, 2008)
    - (Broom & Richtář: Game-Theor. Models in Biology, 2013)

Measure Valued Conservation Laws

(in collaboration with P.Gwiazda & M.Rosińska)

Measure Valued Conservation Laws

$$\begin{cases} \partial_t \mu + \partial_a \big(g(t,\mu)\,\mu\big) + d(t,\mu)\,\mu = 0 \\ \big(g(t,\mu)\big)(0)\; D_\lambda \mu(0+) = \displaystyle\int_{\mathbb{R}^+} b(t,\mu)\;\mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

Measure Valued Conservation Laws

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu)\,\mu\right) + d(t,\mu)\,\mu = 0 \\ \left(g(t,\mu)\right)(0)\; D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu)\,\mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

(Di Perna: ARMA, 1985)

(Ackleh, Ito: JDE, 2004)

(Carrillo, Colombo, Gwiazda & Ulikowska: JDE, 2012)

(Gwiazda, Jamróz & Marciniak-Czochra: SIAM Math.An., 2012)

(Ulikowska: Kinet.Relat.Mod., 2012)

(Canizo, Carrillo & Cuadrado: Acta Appl.Math., 2013)

(Piccoli & Rossi: ARMA, 2014)

Measure Valued Conservation Laws

Escalator Boxcar Train

(de Roos: N.Meth.PDEs, 1988)

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu)\, \mu \right) + d(t,\mu)\, \mu = 0 \\ \left(g(t,\mu) \right)(0)\ D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu)\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

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On $[0, \Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=\mathbf{0}}^{\mathbf{n}} m^i(t)\, \delta_{x^i(t)}$

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On $[0, \Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=\mathbf{0}}^{\mathbf{n}} m^i(t)\, \delta_{x^i(t)}$ where

$$\begin{cases} \dot{x}^i = \Big(g\left(t, \mu^{\mathbf{n}}(t)\right) \Big) (x^i) & i = \mathbf{0}, \ldots, \mathbf{n} \\ \dot{m}^0 = -d\left(t, \mu^{\mathbf{n}}(t)\right)(x^0)\ m^0 + \sum_{i=\mathbf{1}}^{\mathbf{n}} b\left(t, \mu^{\mathbf{n}}(t)\right)(x^i)\ m^i & \\ \dot{m}^i = -d\left(t, \mu^{\mathbf{n}}(t)\right)(x^i)\ m^i & i = \mathbf{1}, \ldots, \mathbf{n} \\ x^i(0) = x^i_0 & i = \mathbf{0}, \ldots, \mathbf{n} \\ m^i(0) = m^i_0 & i = \mathbf{1}, \ldots, \mathbf{n} \\ x^0(0) = 0 & \\ m^0(0) = 0 & \end{cases}$$

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On $[\mathbf{k}\,\Delta t, (\mathbf{k}+1)\Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=-\mathbf{k}}^{\mathbf{n}} m^i(t)\;\delta_{x^i(t)}$

Measure Valued Conservation Laws

Escalator Boxcar Train

(de Roos: N.Meth.PDEs, 1988)

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On $[\mathbf{k}\,\Delta t, (\mathbf{k}+1)\Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=-\mathbf{k}}^{\mathbf{n}} m^i(t)\,\delta_{x^i(t)}$

$$\begin{cases} \dot x^i = \Big(g\left(t,\mu^{\mathbf{n}}(t)\right)\Big)(x^i) & i = -\mathbf{k},\ldots,\mathbf{n} \\ \dot m^{-k} = -d\left(t,\mu^{\mathbf{n}}(t)\right)(x^0)\; m^0 + \sum_{i=-\mathbf{k}+1}^{\mathbf{n}} b\left(t,\mu^{\mathbf{n}}(t)\right)(x^i)\; m^i & \\ \dot m^i = -d\left(t,\mu^{\mathbf{n}}(t)\right)(x^i)\; m^i & i = -\mathbf{k}+1,\ldots,\mathbf{n} \\ x^i(k\,\Delta t) = \lim_{t\to k\Delta t-} x^i_k(t) & i = -\mathbf{k}+1,\ldots,\mathbf{n} \\ m^i(k\,\Delta t) = \lim_{t\to k\Delta t-} m^i_k(t) & i = -\mathbf{k}+1,\ldots,\mathbf{n} \\ x^{-k}(k\,\Delta t) = 0 & \\ m^{-k}(k\,\Delta t) = 0 & \end{cases}$$

Measure Valued Conservation Laws

Escalator Boxcar Train

(de Roos: N.Meth.PDEs, 1988)

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu)\,\mu\right) + d(t,\mu)\,\mu = 0 \\ \left(g(t,\mu)\right)(0)\; D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu)\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

On $[\mathbf{k}\,\Delta t, (\mathbf{k}+1)\Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=-\mathbf{k}}^{\mathbf{n}} m^i(t)\, \delta_{x^i(t)}$

$$\boxed{d\left(\mu(t), \mu^n(t)\right) \leq C \cdot \Big(\Delta t + d\left(\mu_o, \mu^n(0)\right)\Big)}$$

where d is the *flat distance*

$$d(\mu', \mu'') = \sup \left\{ \int_{\mathbb{R}^+} \varphi \, \mathrm{d}(\mu' - \mu'') : \begin{array}{l} \varphi \in \mathbf{C}^1(\mathbb{R}^+; [-1,1]) \\ \|\varphi\|_{\mathbf{W}^{1,\infty}} \leq 1 \end{array} \right\}$$

Measure Valued Conservation Laws

Escalator Boxcar Train

(de Roos: N.Meth.PDEs, 1988)

$$\begin{cases} \partial_t \mu + \partial_a \big(g(t,\mu)\,\mu\big) + d(t,\mu)\,\mu = 0 \\ \big(g(t,\mu)\big)(0)\; D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu)\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

On $[\mathbf{k}\,\Delta t, (\mathbf{k}+1)\Delta t]$ approximate μ with $\mu^{\mathbf{n}}(t) = \sum_{i=-\mathbf{k}}^{\mathbf{n}} m^i(t)\,\delta_{x^i(t)}$

$$\boxed{d\left(\mu(t), \mu^n(t)\right) \le C \cdot \Big(\Delta t + d\left(\mu_o, \mu^n(0)\right)\Big)}$$

(Brännström, Carlsson, & Simpson: SINUM, 2013)

(Colombo, Gwiazda & Rosińka: Preprint, 2016)

Measure Valued Conservation Laws

$$\partial_t \mu + \partial_a (g\, \mu) + d\, \mu = 0 \qquad \longrightarrow \qquad \text{ODE}$$

$$\downarrow$$

$$\mu^n(t) = \sum_{i=-k}^{n} m^i(t)\, \delta_{x^i(t)} \qquad \downarrow$$

$$\downarrow$$

$$\text{solution} \qquad \underset{n \to +\infty}{\longleftarrow} \qquad \text{solution}$$

Measure Valued Conservation Laws – with Control

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu;\mathbf{u})\,\mu \right) + d(t,\mu;\mathbf{u})\,\mu = 0 \\ \left(g(t,\mu;\mathbf{u}) \right)(0)\, D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu;\mathbf{u})\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

Measure Valued Conservation Laws – with Control

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t, \mu; \mathbf{u})\, \mu \right) + d(t, \mu; \mathbf{u})\, \mu = 0 \\ \left(g(t, \mu; \mathbf{u}) \right)(0)\, D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t, \mu; \mathbf{u})\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

On $[0, \Delta t]$ approximate μ with $\mu^n(t) = \sum_{i=1}^{n} m^i(t)\, \delta_{x^i(t)}$ where

$$\begin{cases} \dot{x}^i = \left(g\left(t, \mu^n(t); \mathbf{u(t)}\right) \right)(x^i) & i = 0, \dots, n \\ \dot{m}^0 = -d\left(t, \mu^n(t); \mathbf{u(t)}\right)(x^0)\, m^0 + \sum_{i=1}^{n} b\left(t, \mu^n(t); \mathbf{u(t)}\right)(x^i)\, m^i & \\ \dot{m}^i = -d\left(t, \mu^n(t); \mathbf{u(t)}\right)(x^i)\, m^i & i = 1, \dots, n \\ x^i(0) = x_0^i & i = 0, \dots, n \\ m^0(0) = 0 & \\ m^i(0) = m_0^i & i = 1, \dots, n \end{cases}$$

Measure Valued Conservation Laws – with Control

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu;\mathbf{u})\,\mu \right) + d(t,\mu;\mathbf{u})\,\mu = 0 \\ \left(g(t,\mu;\mathbf{u}) \right)(0)\, D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu;\mathbf{u})\, \mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

Find $\mathbf{u = u(t)}$ to maximize $\mathcal{J}(u) = \int_{\mathbb{R}^+} j\left(t, u(t), \mu(t)\right) \mathrm{d}t$:

Measure Valued Conservation Laws – with Control

$$\begin{cases} \partial_t \mu + \partial_a \left(g(t,\mu;\mathbf{u})\,\mu\right) + d(t,\mu;\mathbf{u})\,\mu = 0 \\ \left(g(t,\mu;\mathbf{u})\right)(0)\,D_\lambda \mu(0+) = \int_{\mathbb{R}^+} b(t,\mu;\mathbf{u})\,\mathrm{d}\mu \\ \mu(0) = \mu_o \end{cases}$$

Find $\mathbf{u = u(t)}$ to maximize $\mathcal{J}(u) = \int_{\mathbb{R}^+} j\left(t, u(t), \mu(t)\right) \mathrm{d}t$:

1. Approximate μ with μ^n
2. *Find* u^n maximizing $\mathcal{J}$ restricted to n–Dirac–δs
3. Let $n \to +\infty$

(Colombo, Gwiazda & Rosińka: Preprint, 2016)

Measure Valued Conservation Laws – with Control

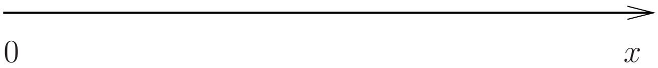
$$
\begin{array}{ccc}
\partial_t \mu + \partial_a (g\mu) = -d\mu & \longrightarrow & \text{ODE} \\
 & & \downarrow \\
 & \mu^n(t) = \displaystyle\sum_{i=-k}^{n} m^i(t)\, \delta_{x^i(t)} & \downarrow \\
 & & \downarrow \\
\text{control} & \underset{n \to +\infty}{\longleftarrow} & \text{control}
\end{array}
$$

Structured Population on Graphs

(in collaboration with M.Garavello)

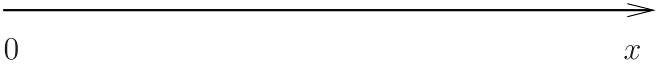
Role of the Graph

The standard setting:

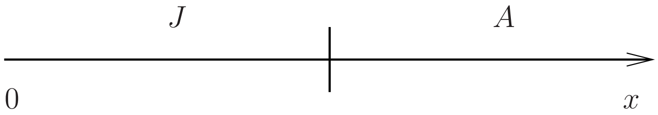


Role of the Graph

The standard setting:



Juvenile – Adult:



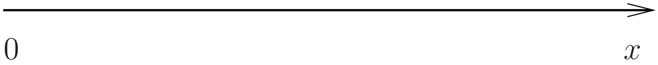
(Carrillo, Cuadrado & Perthame: Math.Biosci., 2007)

(Ackleh & Deng: SIAM Appl.Math., 2009)

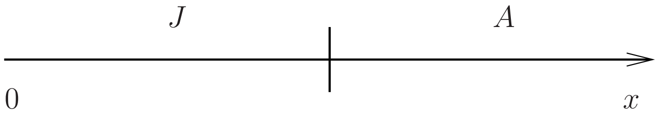
(Ackleh & Ma: Numer.Funct.Anal.Opti., 2013)

Role of the Graph

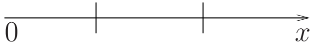
The standard setting:

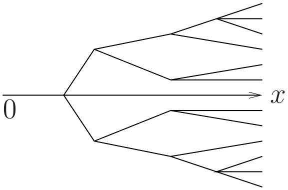


Juvenile – Adult:



Etc. ...





A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \left(g_i(t,x)\, u_i\right) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\left(t, u_1(t), \ldots, u_n(t)\right) \\ u_i(0,x) = \bar{u}_i(x) \end{cases}$$

$t \in \mathbb{R}^+$	time	g_i	growth
$x \in \mathbb{R}^+$	age	$-d_i$	death
$u_i \in \mathbb{R}^+$	density	$\mathcal{B}_i$	birth/change

A General Theorem

$$\left\{\begin{array}{l}\partial_t u_i + \partial_x \big(g_i(t,x)\, u_i\big) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\big(t, u_1(t), \ldots, u_n(t)\big) \\ u_i(0,x) = \bar u_i(x)\end{array}\right.$$

$$\begin{aligned}\mathcal{B}_i(t, u_1, \ldots, u_n) = {} & \alpha_i\big(t, u_1(\bar x_1 -), \ldots, u_n(\bar x_n -)\big) \\ & + \beta_i\left(\int_{I_1} w_1(x)\, u_1(x)\, \mathrm{d}x\,, \ldots, \int_{I_n} w_n(x)\, u_n(x)\, \mathrm{d}x\right)\end{aligned}$$

u_i lives on $[0, \bar x_i]$
u_i is fertile for $x \in I_i$
w_i fertility of u_i

α_i transmission
β_i natality

A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \big(g_i(t,x)\, u_i\big) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\big(t, u_1(t), \ldots, u_n(t)\big) \\ u_i(0,x) = \bar u_i(x) \end{cases}$$

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Theorem (Colombo & Garavello: MBE, 2015)

If $g_i \in \mathbf{C^1}$, $\inf g_i > 0$, $\sup_t \left[\mathrm{TV}\,\big(g_i(t)\big) + \mathrm{TV}\,\big(\partial_x g_i(t)\big)\right] < +\infty$

$d_i \in (\mathbf{C^1} \cap \mathbf{L^\infty})$, $\sup_t \mathrm{TV}\,\big(d_i(t,\cdot)\big) < +\infty$

$\alpha_i, \beta_i, w_i \in \mathbf{C^{0,1}}$, $\alpha_i(t,0) = 0$, $\beta_i(0) = 0$, $\inf_x w_i > 0$

A General Theorem

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Then: *Existence of a solution*

Uniqueness of the solution

Continuous dependence from the initial data

Stability with respect to α_i, β_i, w_i

A General Theorem

$$\begin{cases} \partial_t u_i + \partial_x \big(g_i(t,x)\, u_i\big) = d_i(t,x)\, u_i \\ g_i(t,0)\, u_i(t,0+) = \mathcal{B}_i\big(t, u_1(t), \ldots, u_n(t)\big) \\ u_i(0,x) = \bar u_i(x) \end{cases}$$

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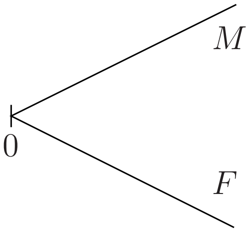
Theorem (Colombo & Garavello: MBE, 2015)

$$\begin{aligned} \left\| u_i'(t) - u_i''(t) \right\|_{\mathbf{L^1}} \le {} & \mathcal{K}(t) \sum_{j=1}^{n} \left[\left\| \bar u_j' - \bar u_j'' \right\|_{\mathbf{L^1}} + t \left\| \bar u_j' - \bar u_j'' \right\|_{\mathbf{L^\infty}} \right] \\ & + \mathcal{H}(t) \sum_{j=1}^{n} \left[\left\| \alpha_j' - \alpha_j'' \right\|_{\mathbf{C^0}} + \left\| \beta_j' - \beta_j'' \right\|_{\mathbf{C^0}} + \left\| w_j' - w_j'' \right\|_{\mathbf{C^0}} \right] \end{aligned}$$

Example: Age & Sex Structured Population



M males F females

(N. Keyfitz: VI Berkeley Symp. Math. Stat. Prob., 1972)

Example: Age & Sex Structured Population

$$\left\{\begin{array}{l}\partial_t M + \partial_a M = -\boldsymbol{\kappa}\,\boldsymbol{\mu}\, M \\ \partial_t F + \partial_a F = -(1-\boldsymbol{\kappa})\,\boldsymbol{\mu}\, F \\ \\ M(t,0) + F(t,0) = \boldsymbol{\nu} \min\left\{\boldsymbol{\vartheta}\int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a\,,(1-\boldsymbol{\vartheta})\int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a\right\} \\ \boldsymbol{\eta}\, M(t,0) = (1-\boldsymbol{\eta})\, F(t,0)\end{array}\right.$$

$\boldsymbol{\eta}$ relative natality in $[0,1]$
$\boldsymbol{\kappa}$ relative mortality in $[0,1]$
$\boldsymbol{\vartheta}$ coupling habits in $[0,1]$

$\boldsymbol{\mu}$ mortality in $\mathbb{R}^+$
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$\kappa = 0$ no Male dies
$\kappa = 1$ no Female dies

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Example: Age & Sex Structured Population

$$
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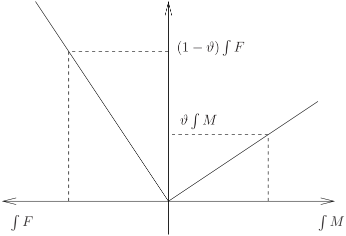

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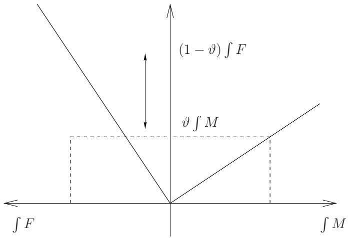
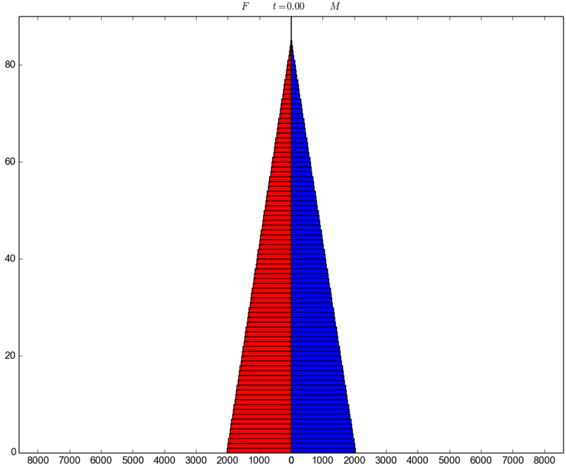


Example: Age & Sex Structured Population

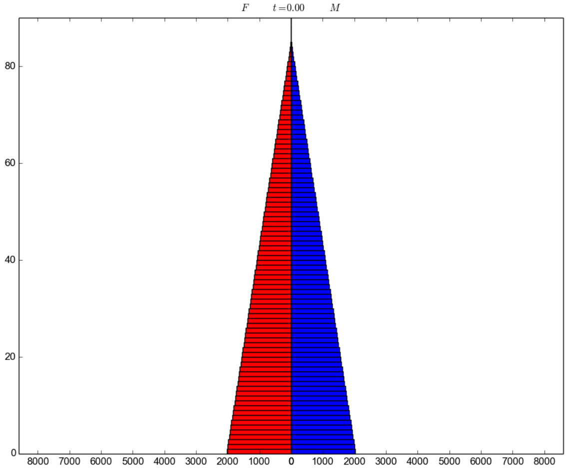
$\kappa = 0.600 \qquad \eta = 0.485 \qquad \vartheta = 0.7067$



Age pyramid - *"slowly"* growing

Example: Age & Sex Structured Population

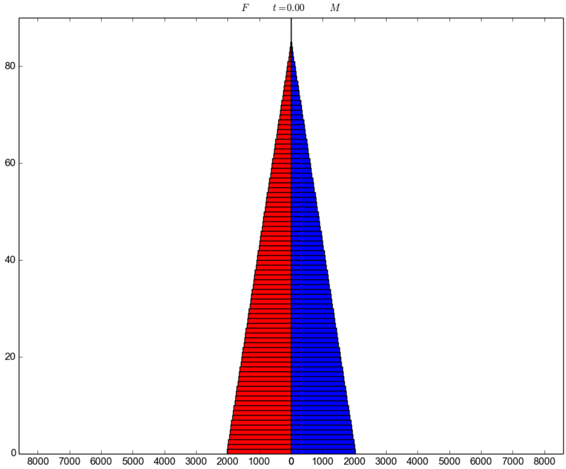
$\kappa = 0.600$ $\eta = 0.300$ $\vartheta = 0.7067$



Lower $\eta \Rightarrow$ too many M are born $\Rightarrow$ extinction

Example: Age & Sex Structured Population

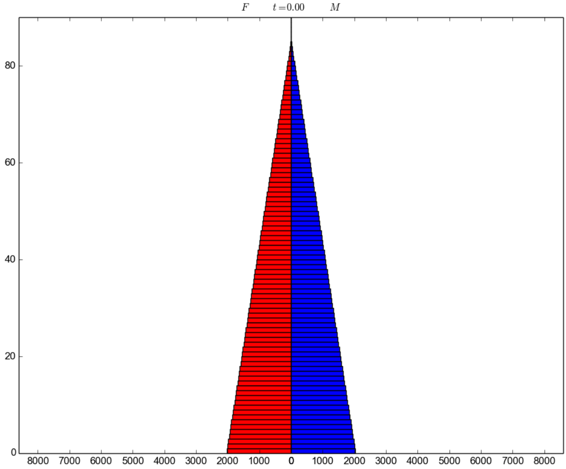
$\kappa = 0.400 \qquad \eta = 0.485 \qquad \vartheta = 0.7067$



Lower $\kappa \Rightarrow$ too many F die $\Rightarrow$ extinction

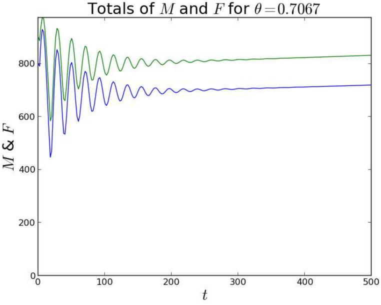
Example: Age & Sex Structured Population

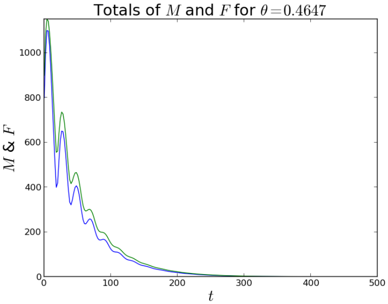
$\kappa = 0.600$ $\eta = 0.485$ $\vartheta = 0.5000$

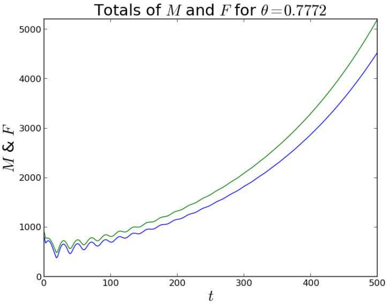


Lower $\vartheta \Rightarrow$ extinction

Example: Age & Sex Structured Population







Example: Age & Sex Structured Population

Optimal Mating Ratio: $$\vartheta(t) = \frac{\int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}{\int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a + \int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}$$

Optimal Fertility Rate: $$\frac{\nu}{\frac{1}{\int_{m_1}^{m_2} M(t,a)\,\mathrm{d}a} + \frac{1}{\int_{f_1}^{f_2} F(t,a)\,\mathrm{d}a}}$$

(Colombo & Garavello: MBE, 2015)

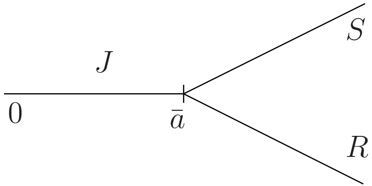
Control

$$\Longleftrightarrow$$

Management of a Biological Resource

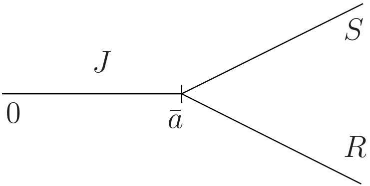
(in collaboration with M. Garavello)

Management of a Biological Resource



J	Juveniles	$a \in [0, \bar{a}]$
S	Sold	$a \in [\bar{a}, +\infty[$
R	Reproduction	$a \in [\bar{a}, a_{\max}]$

Management of a Biological Resource



$$
\begin{cases}
\partial_t J + \partial_a \big(g_J(t,a)\, J\big) = d_J(t,a)\, J \\
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g_J(t,0)\, J(t,0) = \int w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha \\
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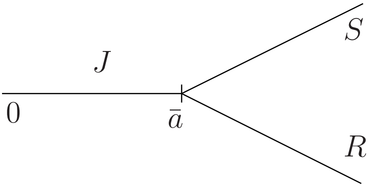
Management of a Biological Resource



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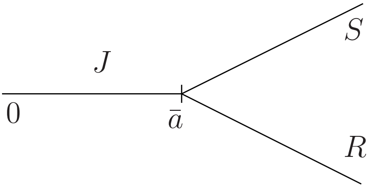
$$\text{Profit} = [\text{income from } S] - [\text{costs of } J, S, R]$$

Management of a Biological Resource



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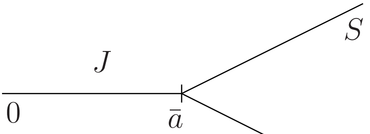
Profit = [income from S] − [costs of J, S, R]

Find $\boldsymbol{\eta}$ to maximize the profit

Stability estimates ⇒ existence of an optimal $\boldsymbol{\eta}$

Management of a Biological Resource



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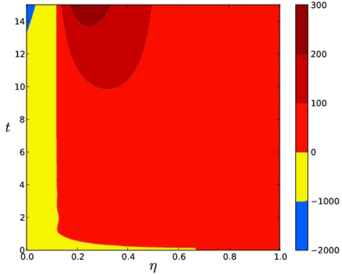
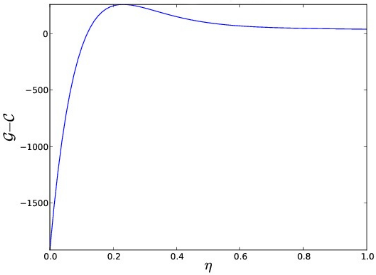
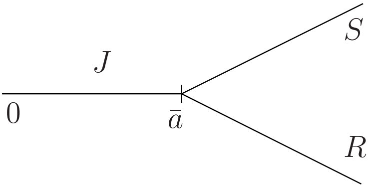




(Colombo & Garavello: MBE, 2015)

Management of a Biological Resource



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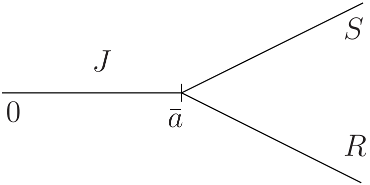
Profit $=$ [income from S] $-$ [costs of J, S, R]

Find $\boldsymbol{\eta}$ to maximize the profit

Stability estimates $\Rightarrow$ existence of an optimal $\boldsymbol{\eta}$

The Profit is Differentiable w.r.t $\boldsymbol{\eta}$!

Management of a Biological Resource



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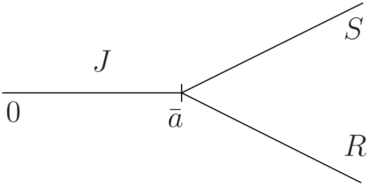
g_R(t,\bar a)\, R(t,\bar a) = \big(1 - \boldsymbol{\eta(t)}\big)\; g_J(t,\bar a)\, J(t,\bar a)
\end{cases}
$$

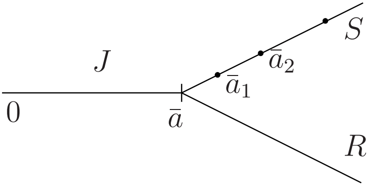
Management of a Biological Resource



$$\begin{cases} \partial_t J + \partial_a \big(g_J(t,a)\, J\big) = d_J(t,a)\, J \\ \partial_t S + \partial_a \big(g_S(t,a)\, S\big) = d_S(t,a)\, S \\ \partial_t R + \partial_a \big(g_R(t,a)\, R\big) = d_R(t,a)\, R \\ g_J(t,0)\, J(t,0) = \int w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha \\ g_S(t,\bar a)\, S(t,\bar a) = \boldsymbol{\eta}(\mathbf{t})\, g_J(t,\bar a)\, J(t,\bar a) \\ g_R(t,\bar a)\, R(t,\bar a) = \big(1 - \boldsymbol{\eta}(\mathbf{t})\big)\, g_J(t,\bar a)\, J(t,\bar a) \end{cases}$$

$$\mathcal{C}_J = \iint C_J\, \big(t, a, J(t,a)\big)\, \mathrm{d}a\, \mathrm{d}t$$
$$\mathcal{C}_S = \iint C_S\, \big(t, a, S(t,a)\big)\, \mathrm{d}a\, \mathrm{d}t$$
$$\mathcal{C}_R = \iint C_R\, \big(t, a, R(t,a)\big)\, \mathrm{d}a\, \mathrm{d}t$$

Management of a Biological Resource



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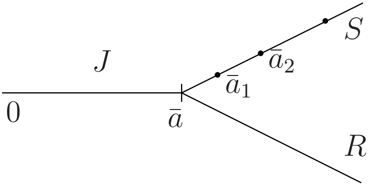
$$\mathcal{C}_J = \iint C_J\,\big(t,a,J(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

$$\mathcal{C}_S = \iint C_S\,\big(t,a,S(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

$$\mathcal{C}_R = \iint C_R\,\big(t,a,R(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta}_{\mathbf{i}}(\mathbf{t})\,P_i(t)\,S(t,\bar a_i-)\,\mathrm{d}t$$

Management of a Biological Resource



$$
\begin{cases}
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\end{cases}
$$

$$\mathcal{C}_J = \iint C_J\,\big(t,a,J(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

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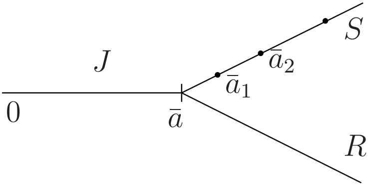
$$\mathcal{C}_R = \iint C_R\,\big(t,a,R(t,a)\big)\,\mathrm{d}a\,\mathrm{d}t$$

$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta_i(t)}\, P_i(t)\, S(t,\bar a_i-)\,\mathrm{d}t$$

$$\mathcal{P} = \mathcal{I} - (\mathcal{C}_J + \mathcal{C}_S + \mathcal{C}_R)$$

Management of a Biological Resource



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$$\mathcal{I} = \int \sum_i \boldsymbol{\vartheta}_{\mathbf{i}}(\mathbf{t})\, P_i(t)\, S(t,\bar a_i-)\,\mathrm{d}t$$
$$\mathcal{P} = \mathcal{I} - (\mathcal{C}_J + \mathcal{C}_S + \mathcal{C}_R)$$

$\mathcal{P}$ is Gateaux differentiable w.r.t. $\boldsymbol{\eta}$ and $\boldsymbol{\vartheta}$

(Colombo & Garavello: ESAIM – COCV, To appear)

Management of a Biological Resource

How to find the optimal control?
maximal profit?

Management of a Biological Resource

How to find the optimal control? / maximal profit?

Gateaux differentiability $\Rightarrow$ gradient methods

Management of a Biological Resource

How to find the optimal control? / maximal profit?

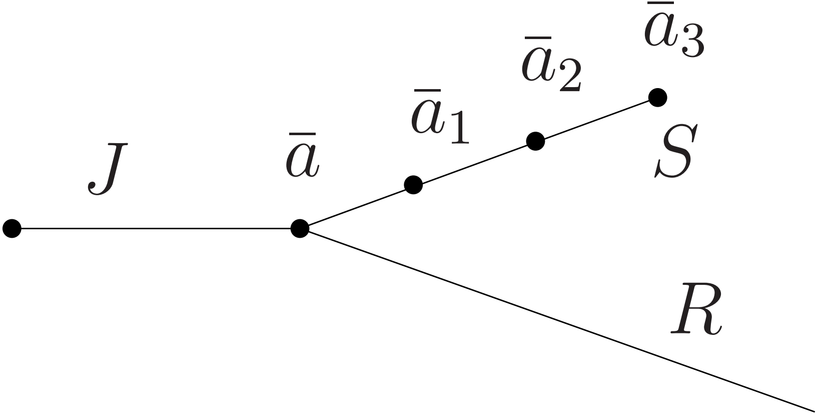
Bang–Bang controls

Theorem (Colombo & Garavello: ESAIM – COCV, To appear)

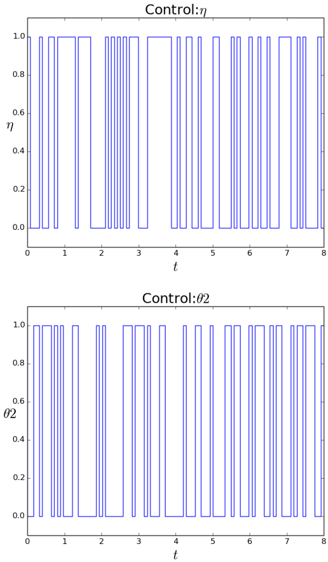
For any $\varepsilon > 0$ there exists a bang–bang control $(\eta_\varepsilon, \vartheta_\varepsilon)$ such that

$$\mathcal{P}(\eta_\varepsilon, \vartheta_\varepsilon) \geq \sup \mathcal{P}(\eta, \vartheta) - \varepsilon$$

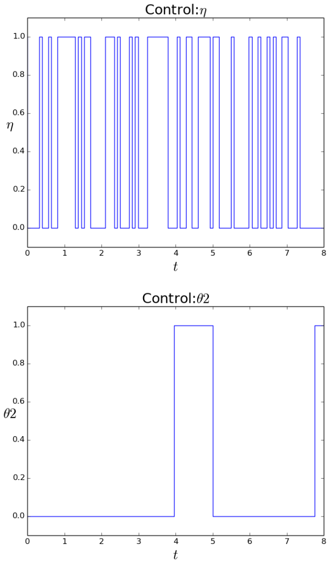
Management of a Biological Resource



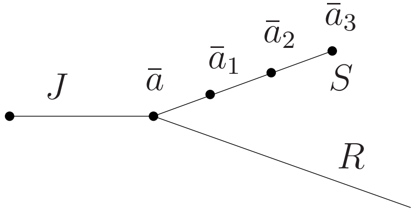
Management of a Biological Resource



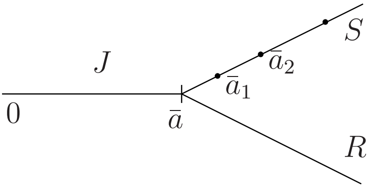
Management of a Biological Resource



Management of a Biological Resource



Management of a Biological Resource



$$
\begin{cases}
\partial_t J + \partial_a \big(g_J(t,a)\, J \big) = d_J(t,a)\, J \\
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g_R(t,\bar{a})\, R(t,\bar{a}) = \big(1 - \boldsymbol{\eta(t)}\big)\, g_J(t,\bar{a})\, J(t,\bar{a})
\end{cases}
$$

$$
\begin{aligned}
\mathcal{C}_J &= \iint c_J(t,a)\, J(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
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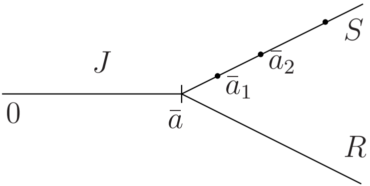
\mathcal{C}_R &= \iint c_R(t,a)\, R(t,a)\, \mathrm{d}a\, \mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{I} &= \int \sum_i \boldsymbol{\vartheta_i(t)}\, P_i(t)\, S(t, \bar{a}_i -)\, \mathrm{d}t \\
\mathcal{P} &= \mathcal{I} - (\mathcal{C}_J + \mathcal{C}_S + \mathcal{C}_R)
\end{aligned}
$$

Theorem (Colombo & Garavello: ESAIM – COCV, To appear)

If $\eta(t) = \sum_i \eta_i\, \chi_{[\tau_{i-1}, \tau_i[}$ *and* $\vartheta(t) = \sum_i \vartheta_i\, \chi_{[\tau_{i-1}, \tau_i[}$, *then*

Management of a Biological Resource



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\begin{cases}
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g_J(t,0)\, J(t,0) = \int w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha\\
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\end{cases}
$$

$$
\begin{aligned}
\mathcal{C}_J &= \iint c_J(t,a)\, J(t,a)\, \mathrm{d}a\, \mathrm{d}t\\
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$$
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Theorem (Colombo & Garavello: ESAIM – COCV, To appear)

If $\eta(t) = \sum_i \eta_i\, \chi_{[\tau_{i-1},\tau_i[}$ *and* $\vartheta(t) = \sum_i \vartheta_i\, \chi_{[\tau_{i-1},\tau_i[}$, *then*

$\mathcal{P}$ *is a polynomial in* $\eta, \vartheta_1, \ldots, \vartheta_n$

(+ bounds on the degree)

Management of a Biological Resource – Example

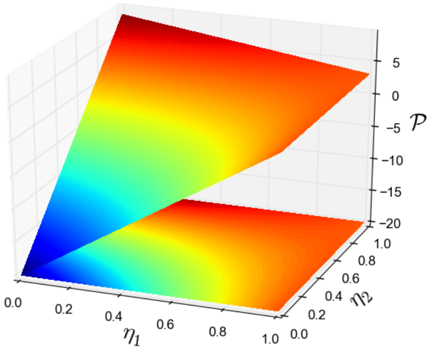
$$
\begin{array}{cccc}
g_J(t,a) = 1.00 & d_J(t,a) = 1.50 & c_J(t,a) = 0.25 & J_o(a) = 1.00 \\
g_S(t,a) = 1.00 & d_S(t,a) = 0.50 & c_S(t,a) = 0.00 & S_o(a) = 0.00 \\
g_r(t,a) = 1.00 & d_R(t,a) = 0.75 & c_R(t,a) = 0.00 & R_o(a) = 0.00 \\
\bar{a} = 1.00 & \bar{a}_1 = 1.50 & N = 1 & \\
p(a) = 0.00 & p_1(t) = 8.00 & w(a) = 120.00\, \chi_{[1.00,\, 4.00]}(a) &
\end{array}
$$

Management of a Biological Resource – Example

$$g_J(t,a) = 1.00 \quad d_J(t,a) = 1.50 \quad c_J(t,a) = 0.25 \quad J_o(a) = 1.00$$
$$g_S(t,a) = 1.00 \quad d_S(t,a) = 0.50 \quad c_S(t,a) = 0.00 \quad S_o(a) = 0.00$$
$$g_r(t,a) = 1.00 \quad d_R(t,a) = 0.75 \quad c_R(t,a) = 0.00 \quad R_o(a) = 0.00$$
$$\bar{a} = 1.00 \quad \bar{a}_1 = 1.50 \quad N = 1$$
$$p(a) = 0.00 \quad p_1(t) = 8.00 \quad w(a) = 120.00\,\chi_{[1.00,\,4.00]}(a)$$

$$\eta = \eta_1\,\chi_{[0,1]} + \eta_2\,\chi_{[1,2]}$$

$$\mathcal{P} = -19.97 + 23.10\,\eta_1 + 28.18\,\eta_2 - 28.18\,\eta_1\,\eta_2$$

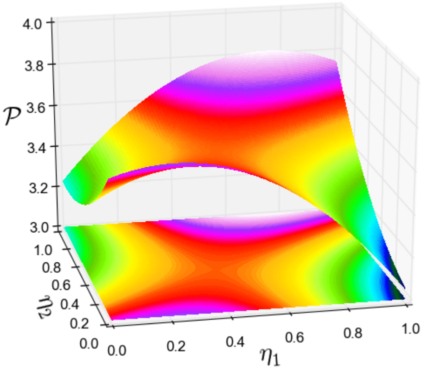


Management of a Biological Resource – Example

$$\bar{a} = 1.00 \quad d_J(t,a) = 0.50 \quad c_J(t,a) = 0.25 \quad p(a) = 1.00 \quad J_o(a) = 1.00$$
$$N = 1 \quad d_S(t,a) = 1.00 \quad c_S(t,a) = 0.25 \quad p_1(t) = 8.20 \quad S_o(a) = 0.00$$
$$\bar{a}_1 = 1.50 \quad d_R(t,a) = 1.50 \quad c_R(t,a) = 0.25 \quad w(a) = 10.00 \quad R_o(a) = 0.00$$

$$\eta = \eta_1 \chi_{[0,0.5]} + \eta_2 \chi_{[0.5,1]} + \eta_1 \chi_{[1,1.5]} + \eta_2 \chi_{[1.5,2]}$$

$$\mathcal{P}(\eta_1, \eta_2) = 3.65 + 0.46\,\eta_1 - 0.88\,\eta_2 + 1.11\,\eta_1\,\eta_2 - 1.06\,{\eta_1}^2 + 0.46\,{\eta_2}^2$$



Management of a Biological Resource

$$
\begin{cases}
\partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J & a \in [0, \bar a] \\
\partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S & a \in [\bar a, +\infty[\\
\partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R & a \in [\bar a, +\infty[\\
g_J(t,0)\, J(t,0) = \int_{\bar a}^{a_{\max}} w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha \\
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g_R(t,\bar a)\, R(t,\bar a) = \left(1 - \boldsymbol{\eta(t)}\right)\, g_J(t,\bar a)\, J(t,\bar a)
\end{cases}
$$

Management of a Biological Resource – PMP

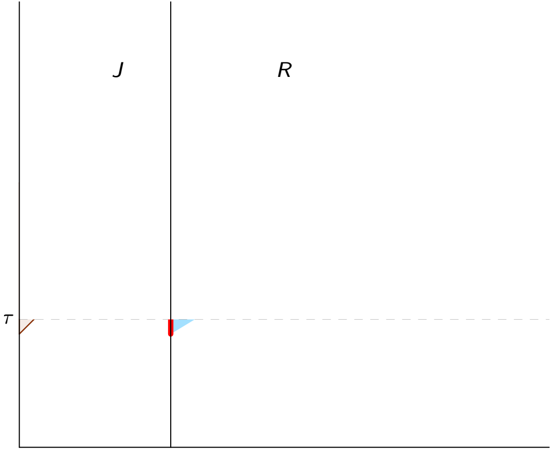
$$\begin{cases} \partial_t J + \partial_a \left(g_J(t,a)\, J\right) = d_J(t,a)\, J & a \in [0, \bar a] \\ \partial_t S + \partial_a \left(g_S(t,a)\, S\right) = d_S(t,a)\, S & a \in [\bar a, +\infty[\\ \partial_t R + \partial_a \left(g_R(t,a)\, R\right) = d_R(t,a)\, R & a \in [\bar a, +\infty[\\ g_J(t,0)\, J(t,0) = \int_{\bar a}^{a_{\max}} w(\alpha)\, R(t,\alpha)\, \mathrm{d}\alpha \\ g_S(t,\bar a)\, S(t,\bar a) = \boldsymbol{\eta(t)}\, g_J(t,\bar a)\, J(t,\bar a) \\ g_R(t,\bar a)\, R(t,\bar a) = \left(1 - \boldsymbol{\eta(t)}\right)\, g_J(t,\bar a)\, J(t,\bar a) \end{cases}$$

The optimal $\boldsymbol{\eta}$ is

$$\boxed{\boldsymbol{\eta_*}(\tau) = \begin{cases} 1 & \text{if } \mathcal{H}(\tau) > 0 \\ 0 & \text{if } \mathcal{H}(\tau) < 0 \end{cases}}$$

PMP – Needle Variations

$$\eta^* = \eta^*(t) \qquad \eta_\varepsilon = \eta_\varepsilon(t) = \begin{cases} \eta^*(t) & \text{if } t \notin [\tau - \varepsilon, \tau] \\ \omega & \text{if } t \in [\tau - \varepsilon, \tau] \end{cases}$$



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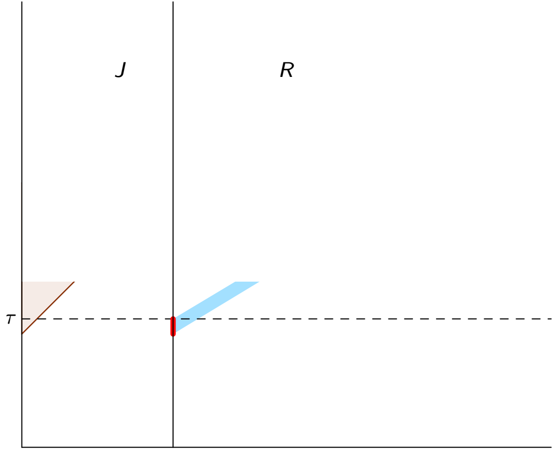

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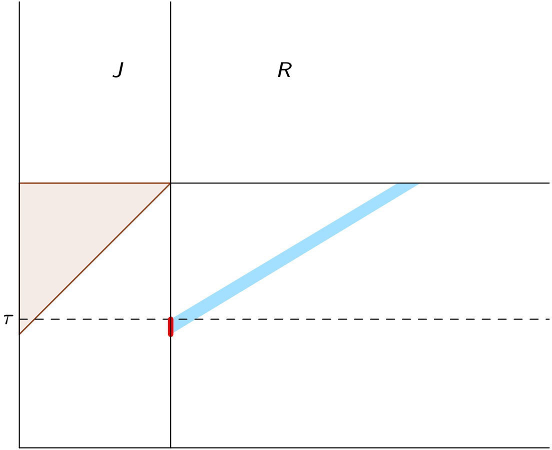

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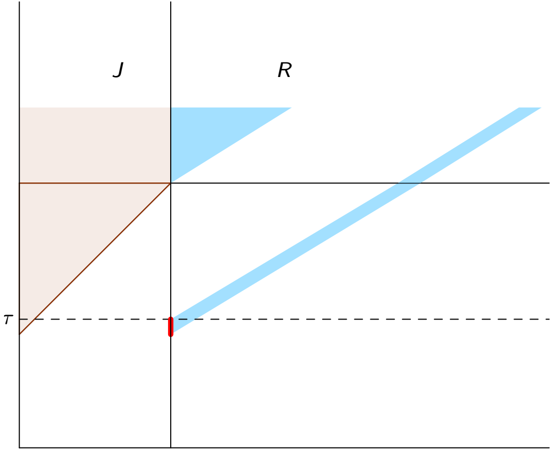

Management of a Biological Resource – PMP

With cost & income

$$
\begin{aligned}
\mathcal{C}(\eta; T) &= \int_0^T \int_0^{\bar a} c_J(t,a)\, J(t,a)\, \mathrm{d}a\, \mathrm{d}t \\
&\quad + \int_0^T \int_{\bar a}^{a_{\max}} \left[c_S(t,a)\, S(t,a) + c_R(t,a)\, R(t,a) \right] \mathrm{d}a\, \mathrm{d}t\,, \\
\mathcal{I}(\eta; T) &= \int_0^T \int_{\bar a}^{a_{\max}} G(t,a)\, S(t,a)\, \mathrm{d}a\ \mathrm{d}t \\
\mathcal{P}(\eta; T) &= \mathcal{I}(\eta; T) - \mathcal{C}(\eta; T)
\end{aligned}
$$

Management of a Biological Resource – PMP

The optimal η is $\boldsymbol{\eta_*}(\tau) = \begin{cases} 1 & \text{if } \mathcal{H}(\tau) > 0 \\ 0 & \text{if } \mathcal{H}(\tau) < 0 \end{cases}$ where

$$
\begin{aligned}
\mathcal{H}(\tau) \;=\; & \int_0^{\mathbf{t_1}} \int_0^{\mathcal{A}_J(t;\tau,0)} \frac{c_J(t,a)}{g_J\left(\mathcal{T}_J(0;t,a),0\right)} w\left(\mathcal{A}_R\left(\mathcal{T}_J(0;t,a);\tau,\bar{a}\right)\right) \\
& \times \exp\left(\int_\tau^{\mathcal{T}_J(0;t,a)} d_R\left(s;\mathcal{A}_R(s;\tau,\alpha)\right) \mathrm{d}s\right) \\
& \times \exp\left(\int_{\mathcal{T}_J(0;t,a)}^{t} \left(d_J\left(s,\mathcal{A}_J(s;t,a)\right) - \partial_a g_J\left(s,\mathcal{A}_J(s;t,a)\right)\right) \mathrm{d}s\right) \mathrm{d}a\,\mathrm{d}t \\
+ & \int_0^{\mathbf{t_1}} \left(G\left(t,\mathcal{A}_S(t;\tau,\bar{a})\right) - c_S\left(t,\mathcal{A}_S(t;\tau,\bar{a})\right)\right) \\
& \times \exp\left(\int_\tau^t d_S\left(s,\mathcal{A}_S(s;\tau,\bar{a})\right) \mathrm{d}s\right) \mathrm{d}t \\
+ & \int_0^{\mathbf{t_1}} c_R\left(t,\mathcal{A}_R(t;\tau,\bar{a})\right) \exp\left(\int_\tau^t d_R\left(s,\mathcal{A}_R(s;\tau,\bar{a})\right) \mathrm{d}s\right) \mathrm{d}t
\end{aligned}
$$

R.M. Colombo, P. Gwiazda, M. Rosinska
Optimization in Structured Population Models through the EBT
Preprint, 2016

R.M. Colombo, M. Garavello
Polynomial Profit in Renewable Resources Management
Work in progress, 2016

R.M. Colombo, M. Garavello
Control of Biological Resources on Graphs
To appear on ESAIM – COCV

R.M. Colombo, M. Garavello
Stability and Optimization in Structured Population Models on Graphs
Mathematical Biosciences and Engineering, 12, 2, 311-335, 2015