Convergence of the solutions in the ergodic approximation of the HJ equation

A. Davini
Sapienza Università di Roma

in collaboration with A. Fathi, R. Iturriaga, M. Zavidovique

Rennes, 30th May 2016

The problem

M compact and connected Riemannian manifold

$$H : T^*M \to \mathbb{R} \qquad \text{continuous}$$

$$\text{(coercivity)} \qquad \inf_{x \in M} H(x,p) \underset{|p|_x \to +\infty}{\longrightarrow} +\infty.$$

The problem

M compact and connected Riemannian manifold

$$H : T^*M \to \mathbb{R} \qquad \text{continuous}$$

$$\text{(coercivity)} \qquad \inf_{x\in M} H(x,p) \underset{|p|_x \to +\infty}{\longrightarrow} +\infty.$$

There exists a unique $c \in \mathbb{R}$ such that the equation

$$H(x, d_x u) = c \qquad \text{in } M \tag{E}$$

admits (viscosity) solutions. The value c is termed **critical**.

The problem

M compact and connected Riemannian manifold

$$H : T^*M \to \mathbb{R} \qquad \text{continuous}$$

$$\text{(coercivity)} \qquad \inf_{x \in M} H(x,p) \underset{|p|_x \to +\infty}{\longrightarrow} +\infty.$$

There exists a unique $c \in \mathbb{R}$ such that the equation

$$H(x, d_x u) = c \qquad \text{in } M \tag{E}$$

admits (viscosity) solutions. The value c is termed **critical**.

[LPV] P.-L. LIONS, G. PAPANICOULAU, S. VARHADAN, *Homogenization of Hamilton-Jacobi equation* (1987).

$(M = \mathbb{R}^N/\mathbb{Z}^N)$

Ergodic approximation

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + H(x, d_x u) = 0 \qquad \text{in } M. \tag{E_λ}$$

Ergodic approximation

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + H(x, d_x u) = 0 \qquad \text{in } M. \tag{E$_\lambda$}$$

Fact: $u, v \in \mathrm{C}(M)$, u super–sol., v sub–sol. $\quad \Rightarrow \quad v \leq u$ in M.
As a consequence, $\exists!\ w_\lambda \in \mathrm{C}(M)$ solution of (E_λ).

Ergodic approximation

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + H(x, d_x u) = 0 \qquad \text{in } M. \tag{E$_\lambda$}$$

Fact: $u, v \in \mathrm{C}(M)$, u super–sol., v sub–sol. $\quad \Rightarrow \quad v \leq u$ in M.
As a consequence, $\exists!\ w_\lambda \in \mathrm{C}(M)$ solution of (E$_\lambda$).

Fact: $\{\lambda w_\lambda\}_{\lambda>0}$ are equi–bounded, $\{w_\lambda\}_{\lambda>0}$ are equi–Lipschitz.

Ergodic approximation

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + H(x, d_x u) = 0 \qquad \text{in } M. \tag{E$_\lambda$}$$

Fact: $u, v \in \mathrm{C}(M)$, u super–sol., v sub–sol. $\quad \Rightarrow \quad v \leq u$ in M.
As a consequence, $\exists!\ w_\lambda \in \mathrm{C}(M)$ solution of (E$_\lambda$).

Fact: $\{\lambda w_\lambda\}_{\lambda>0}$ are equi–bounded, $\{w_\lambda\}_{\lambda>0}$ are equi–Lipschitz.

Moreover, $\quad -\lambda w_\lambda \rightrightarrows c \quad$ in M $\quad$ as $\lambda \to 0^+$.

Ergodic approximation

Let $\lambda > 0$ and consider the equation

$$\lambda u(x) + H(x, d_x u) = 0 \qquad \text{in } M. \tag{E_λ}$$

Fact: $u, v \in \mathrm{C}(M)$, u super–sol., v sub–sol. $\quad \Rightarrow \quad v \leq u$ in M.
As a consequence, $\exists!\ w_\lambda \in \mathrm{C}(M)$ solution of (E_λ).

Fact: $\{\lambda w_\lambda\}_{\lambda>0}$ are equi–bounded, $\{w_\lambda\}_{\lambda>0}$ are equi–Lipschitz.

Moreover, $\quad -\lambda w_\lambda \rightrightarrows c \quad$ in $M \quad$ as $\lambda \to 0^+$.

The functions. $\{\hat{w}_\lambda := w_\lambda - \min_M w_\lambda \,:\, \lambda > 0\}$ are equi–bounded and solve

$$H(x, d_x \hat{w}_\lambda) = -\lambda w_\lambda(x) \qquad \text{in } M.$$

The functions $\{\hat{w}_\lambda\}_{\lambda>0}$ are equi–bounded and equi–Lipschitz and solve

$$H(x, d_x \hat{w}_\lambda) = -\lambda w_\lambda(x) \qquad \text{in } M,$$

The functions $\{\hat{w}_\lambda\}_{\lambda>0}$ are equi–bounded and equi–Lipschitz and solve

$$H(x, d_x \hat{w}_\lambda) = -\lambda w_\lambda(x) \qquad \text{in } M,$$

hence, by the Ascoli–Arzelá Theorem and the stability of the notion of viscosity solution, the $\hat{w}_\lambda$ uniformly converge, along subsequences as $\lambda \to 0^+$, to global viscosity solutions of

$$H(x, d_x u) = c \qquad \text{in } M.$$

This is enough for homogenization.

The functions $\{\hat{w}_\lambda\}_{\lambda>0}$ are equi–bounded and equi–Lipschitz and solve

$$H(x, d_x \hat{w}_\lambda) = -\lambda w_\lambda(x) \qquad \text{in } M,$$

hence, by the Ascoli–Arzelá Theorem and the stability of the notion of viscosity solution, the $\hat{w}_\lambda$ uniformly converge, along subsequences as $\lambda \to 0^+$, to global viscosity solutions of

$$H(x, d_x u) = c \qquad \text{in } M.$$

This is enough for homogenization.

Question: does $\lim_{\lambda \to 0^+} \hat{w}_\lambda$ exist?

Our main result

Theorem

Let $H \in \mathrm{C}(T^*M)$ *be convex and coercive in the momentum. Denote by* $u_\lambda : M \to \mathbb{R}$ *the unique continuous solution of*

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M. \tag{E$_\lambda$}$$

Our main result

Theorem

Let $H \in \mathrm{C}(T^*M)$ be convex and coercive in the momentum. Denote by $u_\lambda : M \to \mathbb{R}$ the unique continuous solution of

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M. \tag{E_λ}$$

Then the functions u_λ uniformly converge on M to a critical solution u_0.

Our main result

Theorem

Let $H \in \mathrm{C}(T^*M)$ be convex and coercive in the momentum. Denote by $u_\lambda : M \to \mathbb{R}$ the unique continuous solution of

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M. \tag{E_λ}$$

Then the functions u_λ uniformly converge on M to a critical solution u_0.

Remark

In (E_λ) we have replaced 0 by c. With this choice the solutions $\{u_\lambda \,:\, \lambda > 0\}$ are equi–bounded.

Remark

If $a \in \mathbb{R}$ and u_λ^a is the solution to

$$\lambda u_\lambda^a(x) + H(x, d_x u_\lambda^a) = a \qquad \text{in } M$$

Remark

If $a \in \mathbb{R}$ and u_λ^a is the solution to

$$\lambda u_\lambda^a(x) + H(x, d_x u_\lambda^a) = a \qquad \text{in } M$$

then

$$u_\lambda^a = u_\lambda + \frac{a - c}{\lambda}.$$

Remark

If $a \in \mathbb{R}$ and u_λ^a is the solution to

$$\lambda u_\lambda^a(x) + H(x, d_x u_\lambda^a) = a \qquad \text{in } M$$

then

$$u_\lambda^a = u_\lambda + \frac{a-c}{\lambda}.$$

Therefore

- the functions $\{u_\lambda^a \,:\, \lambda > 0\}$ are equi–bounded iff $a = c$;

Remark

If $a \in \mathbb{R}$ and u_λ^a is the solution to

$$\lambda u_\lambda^a(x) + H(x, d_x u_\lambda^a) = a \qquad \text{in } M$$

then

$$u_\lambda^a = u_\lambda + \frac{a-c}{\lambda}.$$

Therefore

- the functions $\{u_\lambda^a \,:\, \lambda > 0\}$ are equi–bounded iff $a = c$;
- $u_\lambda(x) - u_\lambda(y) = u_\lambda^a(x) - u_\lambda^a(y)$ for every $x, y \in M$.

Remark

If $a \in \mathbb{R}$ and u_λ^a is the solution to

$$\lambda u_\lambda^a(x) + H(x, d_x u_\lambda^a) = a \qquad \text{in } M$$

then

$$u_\lambda^a = u_\lambda + \frac{a-c}{\lambda}.$$

Therefore

- the functions $\{ u_\lambda^a : \lambda > 0 \}$ are equi–bounded iff $a = c$;
- $u_\lambda(x) - u_\lambda(y) = u_\lambda^a(x) - u_\lambda^a(y)$ for every $x, y \in M$.

In particular, $u_\lambda - \min_M u_\lambda = u_\lambda^a - \min_M u_\lambda^a$ in M.

Corollary

Let $H \in \mathrm{C}(T^*M)$ be convex and coercive in the momentum. Denote by $w_\lambda : M \to \mathbb{R}$ the unique continuous solution of

$$\lambda w_\lambda(x) + H(x, d_x w_\lambda) = 0 \qquad \text{in } M$$

and set $\hat{w}_\lambda := w_\lambda - \min_M w_\lambda$.

Corollary

Let $H \in \mathrm{C}(T^*M)$ be convex and coercive in the momentum. Denote by $w_\lambda : M \to \mathbb{R}$ the unique continuous solution of

$$\lambda w_\lambda(x) + H(x, d_x w_\lambda) = 0 \qquad \text{in } M$$

and set $\hat{w}_\lambda := w_\lambda - \min_M w_\lambda$. Then $\hat{w}_\lambda = u_\lambda - \min_M u_\lambda$, in particular

$$\hat{w}_\lambda \underset{\lambda \to 0^+}{\rightrightarrows} u_0 - \min_M u_0 \qquad \text{in } M.$$

Corollary

Let $H \in \mathrm{C}(T^*M)$ be convex and coercive in the momentum. Denote by $w_\lambda : M \to \mathbb{R}$ the unique continuous solution of

$$\lambda w_\lambda(x) + H(x, d_x w_\lambda) = 0 \qquad \text{in } M$$

and set $\hat{w}_\lambda := w_\lambda - \min_M w_\lambda$. Then $\hat{w}_\lambda = u_\lambda - \min_M u_\lambda$, in particular

$$\hat{w}_\lambda \underset{\lambda \to 0^+}{\rightrightarrows} u_0 - \min_M u_0 \qquad \text{in } M.$$

In the non convex case we have the following partial result:

Proposition

Let $H \in \mathrm{C}(T^*M)$ be coercive in the momentum and assume that the constant functions are critical subsolutions.

Corollary

Let $H \in \mathrm{C}(T^*M)$ *be convex and coercive in the momentum. Denote by* $w_\lambda : M \to \mathbb{R}$ *the unique continuous solution of*

$$\lambda w_\lambda(x) + H(x, d_x w_\lambda) = 0 \qquad \text{in } M$$

and set $\hat{w}_\lambda := w_\lambda - \min_M w_\lambda$*. Then* $\hat{w}_\lambda = u_\lambda - \min_M u_\lambda$*, in particular*

$$\hat{w}_\lambda \underset{\lambda \to 0^+}{\rightrightarrows} u_0 - \min_M u_0 \qquad \text{in } M.$$

In the non convex case we have the following partial result:

Proposition

Let $H \in \mathrm{C}(T^*M)$ *be coercive in the momentum and assume that the constant functions are critical subsolutions.Then* $u_\lambda \geq 0$ *on* M *for every* $\lambda > 0$*, and* $u_\lambda \nearrow u_0$ *uniformly as* $\lambda \searrow 0$ *for some critical solution* u_0*.*

Related literature

Related literature

- M.I. Freidlin, A.D. Wentzell, 1970s
- S. Kamin, 1978, 1980
- W.H. Fleming, 1978
- L.C. Evans, H. Ishii, 1985
- W.H. Fleming, P. Souganidis, 1986
- B. Perthame, 1990
- A. Eizenberg, 1990
- H. Ishii, S. Koike, 1991
- F. Camilli, A. Cesaroni, A. Siconolfi, 2009

Literature

[CDM] I. CAPUZZO–DOLCETTA, J. MENALDI, J. Differential Equations (1988).

Literature

[CDM] I. CAPUZZO–DOLCETTA, J. MENALDI, J. Differential Equations (1988).

$$\max\{\lambda u_\lambda - g(x) \cdot Du_\lambda - f(x), u_\lambda(x) - \psi(x)\} = 0 \qquad \text{in } \mathbb{R}^N$$

with $f, \psi \in \mathrm{BUC}(\mathbb{R}^N)$ & $-g \in \mathrm{Lip}(\mathbb{R}^N)$ and strictly monotone.

Literature

[CDM] I. CAPUZZO–DOLCETTA, J. MENALDI, J. Differential Equations (1988).

$$\max\{\lambda u_\lambda - g(x) \cdot Du_\lambda - f(x), u_\lambda(x) - \psi(x)\} = 0 \qquad \text{in } \mathbb{R}^N$$

with $f, \psi \in \mathrm{BUC}(\mathbb{R}^N)$ & $-g \in \mathrm{Lip}(\mathbb{R}^N)$ and strictly monotone.

They prove that

- $-\lambda u_\lambda \underset{\lambda \to 0^+}{\rightrightarrows} \max\{-f(x_0), 0\}$ in $\mathbb{R}^N$;

Literature

[CDM] I. CAPUZZO–DOLCETTA, J. MENALDI, J. Differential Equations (1988).

$$\max\{\lambda u_\lambda - g(x)\cdot Du_\lambda - f(x), u_\lambda(x) - \psi(x)\} = 0 \qquad \text{in } \mathbb{R}^N$$

with $f, \psi \in \mathrm{BUC}(\mathbb{R}^N)$ & $-g \in \mathrm{Lip}(\mathbb{R}^N)$ and strictly monotone.

They prove that

- $-\lambda u_\lambda \underset{\lambda\to 0^+}{\rightrightarrows} \max\{-f(x_0), 0\}$ in $\mathbb{R}^N$;
- if $f(x_0) < 0$, then $u_\lambda(x) - u_\lambda(x_0) \underset{\lambda\to 0^+}{\rightrightarrows} v$ in $\mathbb{R}^N$, with

$$-g(x)\cdot Dv(x) = f(x) - f(x_0) \qquad \text{in } \mathbb{R}^N.$$

Literature

[IS] R. ITURRIAGA, H. SÁNCHEZ-MORGADO, Discrete Contin. Dyn. Syst. Ser. B (2011).

e.g. $H(x,p) = \frac{|p|^2}{2} - f(x)$ $\quad f \in \mathrm{C}^2(M)$ and such that

$\{y \in M \,:\, f(y) = \min_M f\} = \{y_1, \dots, y_k\}$ hyperbolic fixed points

Literature

[AIPM] N. ANANTHARAMAN, R. ITURRIAGA, P. PADILLA, H. SÁNCHEZ-MORGADO, Disc. Cont. Dyn. Sys. Series B, 5 (2005).

$$-\varepsilon \Delta u_\varepsilon + H(x, Du_\varepsilon) = c(\varepsilon) \qquad \text{in } M$$

Literature

[AIPM] N. ANANTHARAMAN, R. ITURRIAGA, P. PADILLA, H. SÁNCHEZ-MORGADO, Disc. Cont. Dyn. Sys. Series B, 5 (2005).

$$-\varepsilon \Delta u_\varepsilon + H(x, Du_\varepsilon) = c(\varepsilon) \qquad \text{in } M$$

A selection principle for the critical equation is proved under some assumptions on the critical equation.

Literature

[AIPM] N. ANANTHARAMAN, R. ITURRIAGA, P. PADILLA, H. SÁNCHEZ-MORGADO, Disc. Cont. Dyn. Sys. Series B, 5 (2005).

$$-\varepsilon\Delta u_\varepsilon + H(x, Du_\varepsilon) = c(\varepsilon) \qquad \text{in } M$$

A selection principle for the critical equation is proved under some assumptions on the critical equation.

E.g., when $\quad H(x,p) = \frac{|p|^2}{2} - f(x), \quad f \in \mathrm{C}^2(M),$ they assume

$$\{ y \in M \,:\, f(y) = \min_M f \,\} = \{ y_1, \ldots, y_k \,\} \quad \text{hyperbolic fixed points}$$

Literature

[AIPM] N. ANANTHARAMAN, R. ITURRIAGA, P. PADILLA, H. SÁNCHEZ-MORGADO, Disc. Cont. Dyn. Sys. Series B, 5 (2005).

$$-\varepsilon \Delta u_\varepsilon + H(x, Du_\varepsilon) = c(\varepsilon) \qquad \text{in } M$$

A selection principle for the critical equation is proved under some assumptions on the critical equation.

E.g., when $\quad H(x,p) = \frac{|p|^2}{2} - f(x), \quad f \in \mathrm{C}^2(M), \quad$ they assume

$$\{ y \in M \, : \, f(y) = \min_M f \, \} = \{ y_1, \ldots, y_k \, \} \quad \text{hyperbolic fixed points}$$

and

$$\sum_{k=1}^{N} \sqrt{\lambda_k(y_1)} < \sum_{k=1}^{N} \sqrt{\lambda_k(y_i)} \qquad \text{for } i \neq 1,$$

where $\lambda_1(y_i), \ldots, \lambda_N(y_i)$ are the eigenvalues of $D^2 f(y_i)$.

Weak KAM Theory

We recall the main features of weak KAM Theory we employ.

Weak KAM Theory

We recall the main features of weak KAM Theory we employ.

$$\text{(convexity)} \qquad H(x,\cdot) \quad \text{is convex in } T_x^* M \quad \forall x \in M;$$

$$\text{(superlinearity)} \qquad \inf_{x\in M} \frac{H(x,p)}{|p|_x} \underset{|p|_x \to +\infty}{\longrightarrow} +\infty.$$

Weak KAM Theory

We recall the main features of weak KAM Theory we employ.

$$\text{(convexity)} \qquad H(x, \cdot) \quad \text{is convex in } T^*_x M \quad \forall x \in M;$$

$$\text{(superlinearity)} \qquad \inf_{x \in M} \frac{H(x,p)}{|p|_x} \underset{|p|_x \to +\infty}{\longrightarrow} +\infty.$$

Tha associated Lagrangian $L : TM \to \mathbb{R}$ is defined as

$$L(x,q) := \sup_{p \in T^*_x M} \langle p, q \rangle_x - H(x,p)$$

and satisfies properties analogous to those stated above for H.

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ$_a$}$$

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ_a}$$

Fact: The following equivalent characterizations of c hold:

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ$_a$}$$

Fact: The following equivalent characterizations of c hold:

(i) c is the unique real value for which (HJ$_a$) admits solutions;

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ$_a$}$$

Fact: The following equivalent characterizations of c hold:

(i) c is the unique real value for which (HJ$_a$) admits solutions;

(ii) $c = \min\{a \in \mathbb{R} : \text{(HJ}_a\text{) admits subsolutions}\}$;

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ$_a$}$$

Fact: The following equivalent characterizations of c hold:

(i) c is the unique real value for which (HJ$_a$) admits solutions;

(ii) $c = \min\{a \in \mathbb{R} \,:\, (\text{HJ}_a) \text{ admits subsolutions}\}$;

(iii) $-c = \min\left\{\int_{TM} L(x, q)\, \mathrm{d}\tilde{\mu} \,:\, \tilde{\mu} \in \mathscr{P}(TM) \text{ closed}\right\}$.

Let $a \in \mathbb{R}$ and consider

$$H(x, d_x u) = a \quad \text{in } M. \tag{HJ$_a$}$$

Fact: The following equivalent characterizations of c hold:

(i) c is the unique real value for which (HJ$_a$) admits solutions;

(ii) $c = \min\{a \in \mathbb{R} \,:\, (\mathrm{HJ}_a) \text{ admits subsolutions}\}$;

(iii) $-c = \min \left\{ \int_{TM} L(x, q)\, \mathrm{d}\tilde{\mu} \,:\, \tilde{\mu} \in \mathscr{P}(TM) \text{ closed} \right\}.$

Definition: A measure $\tilde{\mu} \in \mathscr{P}(TM)$ is termed closed if

(a) $\int_{TM} |q|_x \, \mathrm{d}\tilde{\mu}(x, q) < +\infty$;

(b) $\int_{TM} \langle d_x \varphi, q \rangle_x \, \mathrm{d}\tilde{\mu}(x, q) = 0 \qquad \forall \varphi \in \mathrm{C}^1(M).$

Definition: The set of the Mather measures is defined as

$$\tilde{\mathfrak{M}} := \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L \, \mathrm{d}\tilde{\mu} = -c \right\},$$

Definition: The set of the Mather measures is defined as

$$\tilde{\mathfrak{M}} := \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L \, \mathrm{d}\tilde{\mu} = -c \right\},$$

while the set of the projected Mather measures is

$$\mathfrak{M} := \left\{ \mu \in \mathscr{P}(M) \,:\, \mu = \pi_{\#}\tilde{\mu} \quad \text{for some} \quad \tilde{\mu} \in \tilde{\mathfrak{M}} \right\}$$

where $\quad \pi_{\#}\tilde{\mu}(B) := \tilde{\mu}(\pi^{-1}(B)) \quad \forall B \in \mathscr{B}(M)$

and $\quad \pi : TM \to M$, $(x, q) \mapsto x$.

$$
\begin{aligned}
\tilde{\mathfrak{M}} &:= \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L \, \mathrm{d}\tilde{\mu} = -c \right\} \\
\mathfrak{M} &:= \left\{ \mu \in \mathscr{P}(M) \,:\, \mu = \pi_{\#}\tilde{\mu} \quad \text{for some} \quad \tilde{\mu} \in \tilde{\mathfrak{M}} \right\}
\end{aligned}
$$

$$\tilde{\mathfrak{M}} := \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L \, \mathrm{d}\tilde{\mu} = -c \right\}$$

$$\mathfrak{M} := \left\{ \mu \in \mathscr{P}(M) \,:\, \mu = \pi_{\#}\tilde{\mu} \quad \text{for some} \quad \tilde{\mu} \in \tilde{\mathfrak{M}} \right\}$$

Mather set

$$\tilde{\mathcal{M}} := \overline{\bigcup_{\tilde{\mu} \in \tilde{\mathfrak{M}}} \text{supp}(\tilde{\mu})}$$

projected Mather set

$$\mathcal{M} := \overline{\bigcup_{\mu \in \mathfrak{M}} \text{supp}(\mu)}$$

$$\tilde{\mathfrak{M}} := \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L \, \mathrm{d}\tilde{\mu} = -c \right\}$$

$$\mathfrak{M} := \left\{ \mu \in \mathscr{P}(M) \,:\, \mu = \pi_{\#}\tilde{\mu} \quad \text{for some} \quad \tilde{\mu} \in \tilde{\mathfrak{M}} \right\}$$

Mather set

$$\tilde{\mathcal{M}} := \overline{\bigcup_{\tilde{\mu} \in \tilde{\mathfrak{M}}} \operatorname{supp}(\tilde{\mu})}$$

projected Mather set

$$\mathcal{M} := \overline{\bigcup_{\mu \in \mathfrak{M}} \operatorname{supp}(\mu)}$$

Fact: $\mathcal{M}$ is a uniqueness set for the critical equation, i.e.

u, v critical solutions, $u = v$ on $\mathcal{M}$ $\Rightarrow u = v$ in M.

Asymptotic convergence

The discounted equation is

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M \tag{E_λ}$$

and the critical equation as

$$H(x, d_x u) = c \qquad \text{in } M. \tag{E}$$

Asymptotic convergence

The discounted equation is

$$\lambda u_\lambda(x) + H(x, d_x u_\lambda) = c \qquad \text{in } M \tag{E$_\lambda$}$$

and the critical equation as

$$H(x, d_x u) = c \qquad \text{in } M. \tag{E}$$

Set

$$u_0(x) := \sup\left\{ v(x) \,:\, v \text{ critical subsol.}, \int_M v \,\mathrm{d}\mu \le 0 \quad \forall \mu \in \mathfrak{M} \right\}$$

$$\tilde{\mathfrak{M}} := \left\{ \tilde{\mu} \in \mathscr{P}(TM) \,:\, \tilde{\mu} \text{ closed}, \int_{TM} L(x,q)\,\mathrm{d}\tilde{\mu}(x,q) = -c \right\}$$

Proposition

$$\int_M u_\lambda \, \mathrm{d}\mu \leq 0 \qquad \forall\, \mu \in \mathfrak{M} \text{ and } \forall\, \lambda > 0.$$

In particular, if $u_{\lambda_n} \rightrightarrows u$ *for some* $\lambda_n \to 0^+$*, then* $u \leq u_0$*.*

Proposition

$$\int_M u_\lambda \, \mathrm{d}\mu \le 0 \qquad \forall\, \mu \in \mathfrak{M} \text{ and } \forall\, \lambda > 0.$$

In particular, if $u_{\lambda_n} \rightrightarrows u$ *for some* $\lambda_n \to 0^+$*, then* $u \le u_0$*.*

Step 2: prove that $u_{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+ \;\Rightarrow\; u \ge u_0$.

Proposition

$$\int_M u_\lambda \,\mathrm{d}\mu \le 0 \qquad \forall\, \mu \in \mathfrak{M} \textit{ and } \forall\, \lambda > 0.$$

In particular, if $u_{\lambda_n} \rightrightarrows u$ *for some* $\lambda_n \to 0^+$*, then* $u \le u_0$*.*

Step 2: prove that $u_{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+ \;\Rightarrow\; u \ge u_0$.

The solution of the discounted equation can be represented as

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} \mathrm{e}^{\lambda s}\Big(L(\gamma(s),\dot\gamma(s)) + c\Big)\, ds.$$

Proposition

$$\int_M u_\lambda \, \mathrm{d}\mu \le 0 \qquad \forall\, \mu \in \mathfrak{M} \textit{ and } \forall\, \lambda > 0.$$

In particular, if $u_{\lambda_n} \rightrightarrows u$ *for some* $\lambda_n \to 0^+$*, then* $u \le u_0$*.*

Step 2: prove that $u_{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+ \;\Rightarrow\; u \ge u_0$.

The solution of the discounted equation can be represented as

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} \mathrm{e}^{\lambda s} \Big(L(\gamma(s), \dot\gamma(s)) + c \Big)\, ds.$$

Theorem

For every $x \in M$ *and* $\lambda > 0$*, the above infimum is attained by a Lipschitz curve* $\gamma_x^\lambda : (-\infty, 0] \to M$ *with* $\gamma_x^\lambda(0) = x$*.*

Proposition

$$\int_M u_\lambda \, \mathrm{d}\mu \le 0 \qquad \forall\, \mu \in \mathfrak{M} \text{ and } \forall\, \lambda > 0.$$

In particular, if $u_{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+$, then $u \le u_0$.

Step 2: prove that $u_{\lambda_n} \rightrightarrows u$ for some $\lambda_n \to 0^+ \;\Rightarrow\; u \ge u_0$.

The solution of the discounted equation can be represented as

$$u_\lambda(x) = \inf_{\gamma(0)=x} \int_{-\infty}^{0} \mathrm{e}^{\lambda s}\Big(L(\gamma(s), \dot\gamma(s)) + c\Big)\, ds.$$

Theorem

For every $x \in M$ and $\lambda > 0$, the above infimum is attained by a Lipschitz curve $\gamma_x^\lambda : (-\infty, 0] \to M$ with $\gamma_x^\lambda(0) = x$. Moreover there exists $\alpha > 0$, independent of λ and x, such that $\|\dot\gamma_x^\lambda\|_\infty \le \alpha$.

Idea: For every $\lambda > 0$ and $x \in M$, we define $\tilde{\mu}_x^\lambda \in \mathscr{P}(TM)$ as

$$\int_{TM} f(x,q)\, d\tilde{\mu}_x^\lambda(x,q) := \int_{-\infty}^{0} \frac{\mathrm{d}}{\mathrm{d}s}(e^{\lambda s})\, f(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s))\, ds$$

for every $f \in \mathrm{C}_c(TM)$.

Idea: For every $\lambda > 0$ and $x \in M$, we define $\tilde{\mu}_x^\lambda \in \mathscr{P}(TM)$ as

$$\int_{TM} f(x,q)\, d\tilde{\mu}_x^\lambda(x,q) := \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}(e^{\lambda s})\, f(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s))\, ds$$

for every $f \in \mathrm{C}_c(TM)$.

Proposition

Let v be a critical subsolution. Then for every $\lambda > 0$ and $x \in M$

$$u_\lambda(x) \geq v(x) - \int_{TM} v(y)\, d\tilde{\mu}_x^\lambda(y,q).$$

Idea: For every $\lambda > 0$ and $x \in M$, we define $\tilde{\mu}_x^\lambda \in \mathscr{P}(TM)$ as

$$\int_{TM} f(x,q)\, d\tilde{\mu}_x^\lambda(x,q) := \int_{-\infty}^0 \frac{\mathrm{d}}{\mathrm{d}s}(e^{\lambda s})\, f(\gamma_x^\lambda(s), \dot{\gamma}_x^\lambda(s))\, ds$$

for every $f \in \mathrm{C}_c(TM)$.

Proposition

Let v be a critical subsolution. Then for every $\lambda > 0$ and $x \in M$

$$u_\lambda(x) \geq v(x) - \int_{TM} v(y)\, d\tilde{\mu}_x^\lambda(y,q).$$

Proposition

Let $x \in M$ and $\lambda_n \to 0^+$. Then, up to subsequences,

$$\tilde{\mu}_x^{\lambda_n} \overset{*}{\rightharpoonup} \tilde{\mu}_x \qquad \text{in } \mathscr{P}(TM)$$

for some Mather measure $\tilde{\mu}_x \in \tilde{\mathfrak{M}}$.

Asymptotic convergence

Theorem

$$u_\lambda \rightrightarrows u_0 \quad \textit{in } M \quad \textit{as } \lambda \to 0^+.$$

Asymptotic convergence

Theorem

$$u_\lambda \rightrightarrows u_0 \quad \textit{in } M \quad \textit{as } \lambda \to 0^+.$$

Proposition

Let $H \in \mathrm{C}(T^*M)$ be coercive in the momentum and assume that the constant functions are critical subsolutions.

Asymptotic convergence

Theorem

$$u_\lambda \rightrightarrows u_0 \quad \text{in } M \quad \text{as } \lambda \to 0^+.$$

Proposition

Let $H \in \mathrm{C}(T^*M)$ be coercive in the momentum and assume that the constant functions are critical subsolutions. Then $u_\lambda \geq 0$ on M for every $\lambda > 0$, and $u_\lambda \nearrow u_0$ uniformly as $\lambda \searrow 0$ for some critical solution u_0.

Asymptotic convergence

Theorem

$$u_\lambda \rightrightarrows u_0 \quad \textit{in } M \quad \textit{as } \lambda \to 0^+.$$

Proposition

Let $H \in \mathrm{C}(T^*M)$ be coercive in the momentum and assume that the constant functions are critical subsolutions.
Then $u_\lambda \geq 0$ on M for every $\lambda > 0$, and $u_\lambda \nearrow u_0$ uniformly as $\lambda \searrow 0$ for some critical solution u_0.
When H is convex, u_0 is the unique critical solution such that
$u_0 = 0 \quad$ *on* $\mathcal{M}$.

Proposition

The function $u_0 = \lim_{\lambda \to 0} u_\lambda$ *obtained above can be characterized in either of the following two ways:*

(i) *it is the largest critical subsolution* $u : M \to \mathbb{R}$ *such that* $\int_M u \, d\mu \leq 0$ *for every projected Mather measure* $\mu \in \mathfrak{M}$*;*

Proposition

The function $u_0 = \lim_{\lambda \to 0} u_\lambda$ *obtained above can be characterized in either of the following two ways:*

(i) *it is the largest critical subsolution* $u : M \to \mathbb{R}$ *such that* $\int_M u \, d\mu \leq 0$ *for every projected Mather measure* $\mu \in \mathfrak{M}$*;*

(i) $$u_0(x) = \inf_{\mu \in \mathfrak{M}} \int_M h(y, x) \, d\mu(y) \qquad \text{for every } x \in M$$

where h *is the Peierls barrier.*

Proposition

The function $u_0 = \lim_{\lambda \to 0} u_\lambda$ *obtained above can be characterized in either of the following two ways:*

(i) *it is the largest critical subsolution* $u : M \to \mathbb{R}$ *such that* $\int_M u \, d\mu \le 0$ *for every projected Mather measure* $\mu \in \mathfrak{M}$*;*

(i) $$u_0(x) = \inf_{\mu \in \mathfrak{M}} \int_M h(y,x)\, d\mu(y) \qquad \text{for every } x \in M$$

where h is the Peierls barrier.

$$h_t(y,x) = \inf \left\{ \int_{-t}^{0} \Big(L(\gamma, \dot\gamma) + c \Big)\, ds \ : \gamma(-t) = y,\ \gamma(0) = x \right\}$$

$$h(y,x) = \liminf_{t \to +\infty} h_t(y,x),$$