Solvability of Fractional Dirichlet Problems with Supercritical Gradient Terms.

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Sections

1. **Nonlocal Operators.**
2. Supercritical problem.
3. Log-critical case.

Nonlocal operators

For a nonnegative measurable function $K : \mathbb{R}^N \to \mathbb{R}$ and a function $u : \mathbb{R}^N \to \mathbb{R}$, an elliptic nonlocal operator takes the form

$$\mathcal{I}(u,x) = \int_{\mathbb{R}^N} [u(y) - u(x)] K(x-y) dy,$$

each time the integral has a sense.

- The basic assumptions on K is the Lévy condition

$$\int_{\mathbb{R}^N} \text{mín}\{1, |y|^2\} K(y) dy < \infty.$$

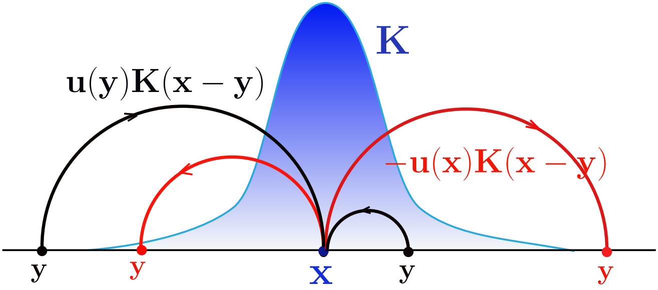
This implies growth at infinity and regularity requirements over u in order $\mathcal{I}(u)$ is well-defined.

Population dynamic interpretation.

When $||K||_{L^1(\mathbb{R}^N)} = 1$, the nonlocal heat equation

$$u_t = \mathcal{I}(u), \quad \text{in } \mathbb{R}^N \times (0, \infty)$$

models the population dynamics driven by the nonlocal displacement of the individuals in the space.



The Fractional Laplacian

The Fractional Laplacian of order σ for $\sigma \in (0,2)$ is given by

$$(-\Delta)^{\sigma/2}u(x) = -C_{N,\sigma}\mathrm{P.V.}\int_{\mathbb{R}^N}\frac{u(y)-u(x)}{|x-y|^{N+\sigma}}dy,$$

where $C_{N,\sigma}>0$ is a normalizing constant making

$$(-\Delta)^{\sigma/2} \to -\Delta \quad \text{as} \quad \sigma \to 2^-.$$

- It is an operator that "differentiates σ-times".
Comparison principle, regularizing effect (uniform ellipticity), variational structure, Fourier transform (hamonic analysis), infinitesimal generator of jump Lévy preocesses,...

- Lack of product rule, chain rule,...

The Supercritical Problem.

For $\Omega \subset \mathbb{R}^N$ bounded with smooth boundary, we are interested in the solvability of nonlocal Dirichlet problems with the form

$$\begin{cases} \lambda u + (-\Delta)^{\sigma/2} u + |Du|^p = f & \text{in } \Omega \\ u = \varphi & \text{in } \Omega^c \end{cases} \qquad \text{(DP)}$$

where $\lambda \in \mathbb{R}$, $f \in C(\bar{\Omega}), \varphi \in C_b(\Omega^c)$.

AIM: Solve (in the viscosity sense) the Dirichlet problem (DP) continuously in $\mathbb{R}^N$ under the supercritical assumption

$$p > \sigma$$

.

Second-order case

Let $\lambda > 0$. For the second-order problem

$$\begin{cases} \lambda u - \Delta u + |Du|^p &= f \qquad \text{in } \Omega \\ \qquad\qquad\qquad u &= \varphi \qquad \text{on } \partial\Omega \end{cases} \tag{1}$$

has a deep connection with stochastic optimal control problems. As it can be seen in Barles & Da Lio [JMPA'04]:

- Problem (1) is uniquely solvable by $u \in C(\bar{\Omega})$ satisfying the equation with generalized boundary conditions.

- If $p \leq 2$ the boundary data is attained classically ($u = \varphi$ on $\partial\Omega$).

- If $p > 2$, loss of the boundary condition phenomena may occur, that is, it may happen that

$$u(x) \neq \varphi(x)$$

at some points $x \in \partial\Omega$.

Second-order case

Capuzzo-Dolcetta, Leoni & Porretta [TAMS'10] prove that the superquadratic equation (1) can be continuously solvable (up to the boundary) provided

- Compatibility condition among f and φ:

$$\lambda \inf_{\partial\Omega}\{\varphi\} \leq \inf_{\bar{\Omega}}\{f\}$$

- Oscillation control of the boundary condition: There exists $M > 0$ small enough such that

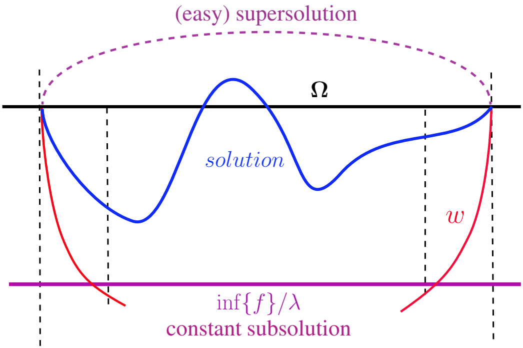
$$|\varphi(x) - \varphi(y)| \leq M|x-y|^{\frac{p-2}{p-1}}, \quad \text{for all } x, y \in \partial\Omega.$$

This last condition is in fact a compatibility condition with the C^γ regularity of **subsolutions** to problem (1) (attaining or not the boundary condition).

The above assumptions allow to construct a function with the “form”

$$w(x) \sim -\tilde{M} d(x)^{\gamma}$$

with $\tilde{M} > 0$ large enough to reach the constant subsolution, BUT not that large in order to have the viscosity inequality near the boundary.



Key: Ellipticity.

Nonlocal problem

For the nonlocal Dirichlet problem (DP) we have several similarities

• Barles, Chasseigne & Imbert [IUJM'08]: If $p \leq \sigma$, problem (DP) has a unique viscosity solution $u \in C(\mathbb{R}^N)$ with $u = \varphi$ on Ω^c.

▶ Explicit examples of loss of the boundary condition when the gradient dominates.

• Barles & T. [SIMA'16]: Existence of a unique viscosity solution to problem (DP) with generalized boundary condition for any $p > 0$.

• Barles, Koike, Ley & T. [CVPDE'14]: When $p > \sigma$, interior $\mathbf{C}^{\frac{p-\sigma}{p-1}}$ estimates hold for problem (DP), and $\mathbf{C}^{\frac{p-\sigma}{\sigma}}$ regularity estimates up to the boundary.

▶ Presence of jump discontinuities is fatal in the nonlocal setting.

Theorem (Dávila, Quaas & T.)

Assume φ *is supported on a set* $D \subset \mathbb{R}^N$ *compact with* $\Omega \subset\subset D$*, and denote*

$$\lambda_0(x) = C_{N,\sigma} \inf_{x \in \Omega} \int_{D^c} \frac{dy}{|x-y|^{N+\sigma}} \in (0, +\infty).$$

Consider $\lambda > -\lambda_0$. *Assume that* $\sigma \in (1,2)$, $p \in (\sigma, \sigma/(2-\sigma))$ *and that* φ, f *satisfy*

$$(\lambda_0 + \lambda) \inf_D \{\varphi\} \le \inf_\Omega \{f\}$$

and

$$|\varphi(x) - \varphi(y)| \le M |x-y|^{\frac{p-\sigma}{p-1}}$$

for all $x, y \in D \setminus \Omega$*, for some* $M > 0$ *small enough.*
Then, there exists a unique viscosity solution $u \in C^{\frac{p-\sigma}{p-1}}(\mathbb{R}^N)$ *to the problem* (DP).

Following CD-L-P's paper:

- For each $y \in \partial\Omega$ it is defined

$$v_y(x) = (\varphi(y) - M|x-y|^{\gamma})\mathbf{1}_D(x) - \mu M d_+^{\gamma}(x),$$

where $\gamma = (p-\sigma)/(p-1)$.

- For suitable $\mu > 0$, it is proven that

$$v = \inf_{y \in \partial\Omega} \{v_y\}$$

is a subsolution near the boundary (and attains the boundary data).

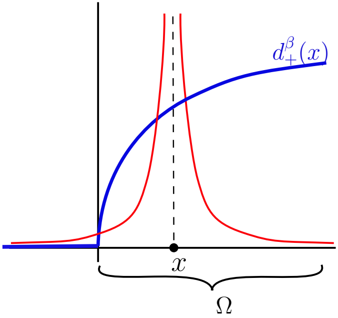
- Using the assumptions it is possible to extend the subsolution to the whole domain by "gluing it"with the constant subsolution.

- Once the barrier with Hölder profile γ is constructed, γ-Hölder regularity up to the boundary for the solution follows standard arguments.

Remarks

- Role of the ellipticity: If $\sigma > 1$ and $\beta < \sigma/2$, then for some $C > 0$ and all $x \in \Omega$ close to the boundary

$$(-\Delta)^{\sigma/2} d_+^{\beta}(x) \leq -C d^{\beta-\sigma}(x).$$



The balance of powers precedure that leads to the result in the proof implies the condition $p < \sigma/(2-\sigma)$.

Remarks

• General φ (not compactly supported): We artificially write $\varphi = \varphi_1 + \varphi_2$ with

$$\varphi_1 = \varphi \mathbf{1}_D \quad \text{and} \quad \varphi_2 = \varphi \mathbf{1}_D^c$$

for some $\Omega \subset\subset D$ where φ is γ-Hölder. We incorporate $(-\Delta)^{\sigma/2}\varphi_2$ to the right-hand side f.

• The result is compatible with the second-order case when we make $\sigma \to 2^-$ (in an adequate way).

• Extension to fully nonlinear nonlocal problems: The upper bound for p is restricted by the ellipticity constants of the nonlocal operator.

Motivation

Alarcón, García-Melián & Quaas [PRSEd'14]; and Kawohl & Kutev [AMSc'12] address the problem

$$\begin{cases} \lambda u - \Delta u &= g(|Du|) + f \qquad \text{in } \Omega \\ u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega \end{cases} \tag{2}$$

where $g \in C^1[0, +\infty) \to \mathbb{R}$ is increasing, coercive and $g(0) = 0$.

They prove that (2) is continuously solvable if g satisfies the condition

$$\int_1^{+\infty} \frac{tdt}{g(t)} = +\infty, \tag{3}$$

allowing nonlinearities g with the asymptotic form $g(t) = t^2 \log^{\alpha}(t)$ for $\alpha < 1$.

Moreover, the solutions have bounded gradient at $\partial\Omega$, leading to classical solutions in several interesting cases.

Main idea

Lemma

Given $M_1, M_2 > 0$ and $\delta > 0$, there exists $L > 0$ large enough such that the ode problem

$$\begin{cases} -h''(t) = g(h'(t)) + M_1 & t \in (0,\delta) \\ h(0) = 0, \ h(\delta) = M_2, \\ h'(0) = L, \ \text{and } h' \geq 0 \text{ in } (0,\delta) \end{cases}$$

has a (unique) solution $h \in C[0,\delta] \cap C^2(0,\delta)$.

In fact, denoting

$$H(z) = H_{L,M_1}(z) = \int_z^L \frac{dy}{g(y) + M_1},$$

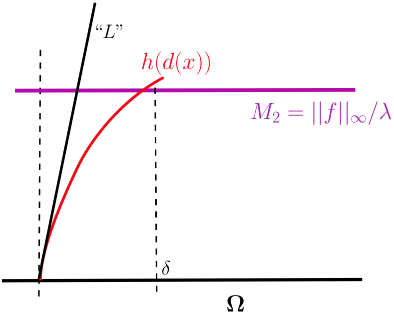
then h has the "explicit" form

$$h(t) = \int_0^t H^{-1}(y)dy.$$

Main idea

Then, it is possible to construct a supersolution to the problem with a function with the form

$$v(x) = h(d(x)), \quad x \in \Omega.$$



- Key: Chain rule (to pass from the pde to the ode).

Fractional to ode

Lemma (Dávila, Quaas & T.)

Let $\sigma \in (1,2)$ and $h : [0,+\infty] \to \mathbb{R}$ a smooth, bounded function with $h' > 0, h'' < 0$ and h'' monotone.
Define $v(x) = h(d_+(x))$. Then, for each $x \in \Omega$ close to the boundary we have

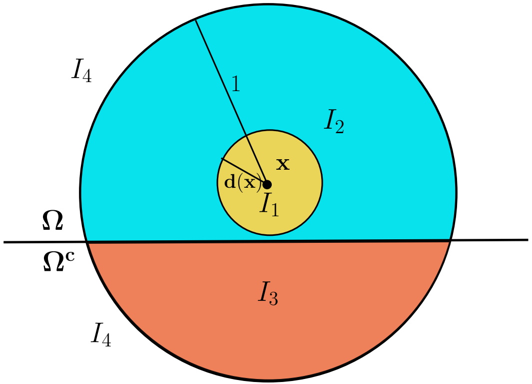
$$(-\Delta)^{\sigma/2} v(x) \geq I_1 + I_2 + I_3 + I_4$$

where for some constants $c, C, C_1, C_2, C_3 > 0$ we denote

$$\begin{aligned} I_1 &:= -\, c d(x)^{2-\sigma} h''(d(x)) - C d(x)^{1-\sigma} h(d(x)) \\ I_2 &:= -\, C_1 d(x)^{1-\sigma} h'(d(x)) \\ I_3 &:= C_2 d^{-\sigma}(x) h(d(x)) \\ I_4 &:= -\, C_3 ||h||_\infty . \end{aligned}$$

The constants $C_i \to 0$ as $\sigma \to 2^-$.

Splitting of the integral



- The key estimate concerning I_1 is a consequence of the "ellipticity in the gradient direction" presented by Barles, Chasseigne & Imbert [JEMS'11].

Ishii-Nakamura operator

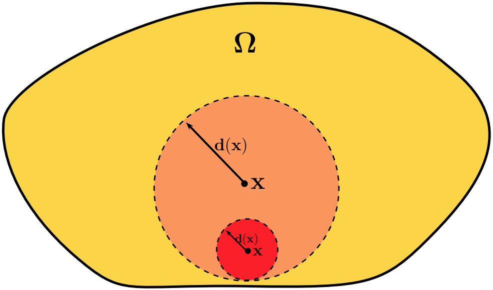
The term coming from I_2 creates serious problems to adapt the ode approach to a problem with the (full) fractional Laplacian.

We present a log-critical result for the Ishii-Nakamura operator [CVPDE'10] defined as

$$\mathcal{I}(u,x) = C_{N,\sigma}\ \mathrm{P.V.} \int_{B(x,d(x))} \frac{u(y)-u(x)}{|x-y|^{N+\sigma}}\,dy,$$

which have shown to be a valid approximation of the Laplacian (and p-Laplacian) as $\sigma \to 2^-$ in bounded domains.

- Ishii-Nakamura operator may be regarded as a degenerate nonlocal operator degenerating in the normal direction to the boundary.



Theorem (Dávila, Quaas & T.)

Given a superlinear nonlinearity g we define the function

$$H(z) = \int_z^{+\infty} \frac{dy}{g(y)}, \quad \text{for } z > 0.$$

Assume g satisfies the condition

$$\int_1^{+\infty} \frac{tH(t)^{\frac{2-\sigma}{\sigma-1}}}{g(t)} dt = +\infty. \tag{4}$$

Then, for all $\lambda > 0$ and $f \in C(\bar{\Omega})$ there exists a unique viscosity solution $u \in C(\bar{\Omega})$ to the problem

$$\begin{cases} \lambda u - \mathcal{I}(u) = g(|Du|) + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{5}$$

which attainsthe boundary data continuously.

Observations

- The function $g(t) = t^\sigma$ satisfies (4) since a direct computation leads to

$$\int_1^{+\infty} \frac{dt}{t} = +\infty.$$

- A function with the form $g(t) = t^\theta$ with $\theta > \sigma$ does not satisfy (4) since in that case the integral has the form

$$\int_1^{+\infty} \frac{dt}{t^{\frac{\theta-1}{\sigma-1}}} < +\infty.$$

- A function with the form $g(t) = t^\sigma \log^\alpha(t)$ with $0 < \alpha < \sigma - 1$ satisfies (4). This is based on the fact that for such a g, the associated function H satisfies the estimate

$$H(t) \geq c t^{1-\sigma} \log^{-\alpha}(t) \quad \text{as } t \to +\infty.$$

- The splitting lemma suggest to solve the ode

$$\begin{cases} -t^{2-\sigma}h''(t) = g(h'(t)) + M_1 & t \in (0,\delta) \\ h(0) = 0, \ h(\delta) = M_2, \\ h'(0) = L, \ \text{and } h' \geq 0 \text{ in } (0,\delta) \end{cases}$$

with L large enough depending on M_1, M_2 and δ.

- In fact, this ode is integrable in $(0,\delta)$ and its solution h satisfies the requirements of the lemma.

- We use $h(d(x))$ as a barrier with

$$M_2 = ||f||_\infty/\lambda \quad \text{and} \quad M_1 = ||f||_\infty + CM_2$$

where $C > 0$ is the constant coming from the splitting lemma.

- Finally, $L >> 1$ is fixed in terms of the data in order to control the extra term

$$Cd^{1-\sigma}(x)h(d(x))$$

coming from the splitting lemma.

Thank you!