Homogenization of a 1D pursuit law with delay

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Traffic flow modelling: microscopic scale

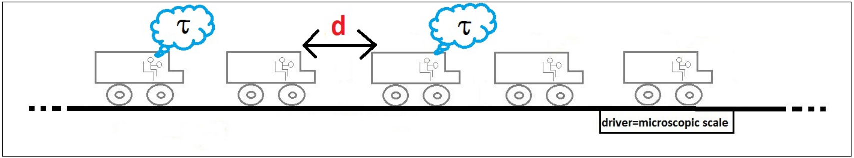

Figure : View of the microsopic scale considered in our study.

Traffic flow modelling: macroscopic scale

Figure : Schematic view of the global traffic flow around Paris from sytadin.fr.

Homogenization process: micro-macro passage

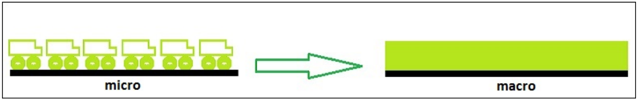

Figure : Schematic description of the homogenization process.

Huge reaction times $\Rightarrow$ accordion effect (see video)

Pursuit law

Microscopic dynamics (DDE):

$$\frac{dX_i}{dt}(t) = F(X_{i+1}(t-\tau) - X_i(t-\tau)) \qquad (i,t) \in \mathbb{Z} \times (0,+\infty)$$

with initial condition:

$$X_i(t) = x_i^0(t) \qquad t \in [-2\tau, 0]$$

Rescaled equation and expected limit

Hyperbolic rescaling:

$$u^{\varepsilon}(x,t) := \varepsilon X_{\lfloor \frac{x}{\varepsilon} \rfloor}(\frac{t}{\varepsilon})$$

Microscopic model embedded into:

$$\begin{cases} \partial_t u^{\varepsilon}(x,t) = F\left(\frac{u^{\varepsilon}(x+\varepsilon,t-\varepsilon\tau)-u^{\varepsilon}(x,t-\varepsilon\tau)}{\varepsilon}\right) & (x,t)\in\mathbb{R}\times(0,+\infty) \\ u^{\varepsilon}(x,s) = u_0(x,s) & (x,s)\in\mathbb{R}\times[-2\varepsilon\tau,0] \end{cases}$$

Expected macroscopic model for vanishing ε:

$$\begin{cases} \partial_t u^0(x,t) = F(\partial_x u^0(x,t)) & (x,t)\in\mathbb{R}\times(0,+\infty) \\ u^0(x,0) = u_0(x,0) & x\in\mathbb{R} \end{cases}$$

Outline

Some references

Microscopic traffic flow models
[Chandler, Hermann, Montroll; (1958)]
[Gipps; (1981)]
[Costeseque; (2014)]
Macroscopic traffic flow models
[Lighthill, Whitham; (1955)]
[Richards; (1956)]
Periodic homogenization for system of ODEs
[Forcadel, Imbert, Monneau; (2007)]
[Forcadel, Imbert, Monneau; (2008)]
Homogenization with a junction condition
[Forcadel, Salazar; (2014)]
Homogenization of second order models
[Forcadel, Salazar; (2015)]

Existence and Uniqueness of solutions

Assumptions

(B0)

$$F \text{ is: } \begin{cases} \text{non-decreasing} \\ \text{bounded} \\ C_F - \text{Lipschitz} \end{cases}$$

(B1) u_0 is $L-$Lipschitz

- Microscopic model ($\varepsilon-$singular DDE): Incremental construction
- Macroscopic model (HJ equation): Perron method and comparison principle

Convergence theorem

Theorem [F-Monneau]

For $\tau \in \left(0; \frac{1}{eC_F}\right)$ and under assumptions (B0)-(B1), u^ε converges locally uniformly towards u^0.

The proof relies on a strict comparison principle stated on:

$$\begin{cases} \partial_t u(x,t) = F(u(x+1,t-\tau) - u(x,t-\tau)) & (x,t) \in \mathbb{R} \times (0,+\infty) \\ u(x,s) = u_0(x,s) & (x,s) \in \mathbb{R} \times [-2\tau; 0] \end{cases}$$

Strict comparison principle

Theorem [F-Monneau]

Under (B0)-(B1), let v be a supersolution and u a subsolution. If:

$$0 < \delta \leq (v-u)(x,t-\tau') \leq \rho(\tau')(v-u)(x,t) \quad \tau' \in [0,\tau], (x,t) \in \mathbb{R}\times[-\tau,0]$$

where ρ is non decreasing and such that:

$$1 + C_F \rho(\tau) \int_0^{\tau'} \rho(s)ds < \rho(\tau') \qquad \tau' \in [0,\tau]$$

Then we have:

$$0 < \delta e^{-C_F \rho(\tau) t} \leq v(x,t) - u(x,t) \qquad (x,t) \in \mathbb{R}\times[0,+\infty)$$

Restriction on τ

Lemma

There exist non decreasing ρ such that:

$$1 + C_F \rho(\tau) \int_0^{\tau'} \rho(s) ds < \rho(\tau') \qquad \tau' \in [0, \tau]$$

if and only if $\tau < \frac{1}{eC_F}$.

The proof uses comparison principles on affine differential inequalities.

The restriction on τ is not sufficient for the strict comparison principle to hold. Vehicles must be suitably spaced out.

Proof of strict comparison principle (sketch)

We want to prove that:

$$0 < \delta e^{-C_F \rho(\tau) t} \leq d(x,t) := v(x,t) - u(x,t) \qquad (x,t) \in \mathbb{R} \times [0,+\infty)$$

$$T^* := \sup\{S \geq 0 / d(x, t-\tau') \leq \rho(\tau') d(x,t), \tau' \in [0,\tau], x \in \mathbb{R}, t \in [-\tau, S]\}$$

We show by contradiction that $T^* = +\infty$.

By definition of d, as F is C_F-Lipschitz and $T^* = +\infty$:

$$\partial_t d(x,t) \geq -C_F d(x, t-\tau) \geq -C_F \rho(\tau) d(x,t)$$

Let us show that $T^* = +\infty$:
By contradiction, if not:

$$\forall \beta \in (0,1), \exists \tau_\beta \in [0,\tau], \exists (x_\beta, t_\beta) \in \mathbb{R} \times (T^*, T^* + \beta] :$$

$$d(x_\beta, t_\beta - \tau_\beta) > \rho(\tau_\beta) d(x_\beta, t_\beta)$$

Preliminary result:

$$d(x, t - \tau') \leq \bar{\rho}(\tau') d(x,t) \qquad \tau' \in [0,\tau], (x,t) \in \mathbb{R} \times [0, T^*]$$

with $\bar{\rho}$ given by:

$$\bar{\rho}(\tau') = 1 + C_F \rho(\tau) \int_0^{\tau'} \rho(s) ds \ \ (< \rho(\tau'))$$

Proof based on classical comparison principle for the ODE:

$$\partial_{\tau'} W = C_F \rho(\tau) \rho(\tau') W$$

Case 1: $t_\beta - \tau_\beta \leq T^*$.

$$d(x_\beta, t_\beta - \tau_\beta) = d(x_\beta, T^* - \tau'_\beta)$$

with $\tau'_\beta := T^* - t_\beta + \tau_\beta \in [0, \tau]$

Use of preliminary result:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \bar{\rho}(\tau'_\beta) d(x_\beta, T^*)$$

Moreover, d satisfies:

$$d(x,t) \geq d(x,s) - 2||F||_\infty (t-s) \qquad (x,t) \in \mathbb{R} \times [0,+\infty), s \in [0,t]$$

Hence:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \bar{\rho}(\tau'_\beta)(d(x_\beta, t_\beta) + 2||F||_\infty \beta)$$

Then, $\beta < \beta_1 := \frac{\delta}{2||F||_\infty + 1} e^{-C_F \rho(\tau) T^*} > 0$ implies:

$$d(x_\beta, t_\beta) \geq d(x_\beta, T^*) - 2||F||_\infty \beta \geq \delta e^{-C_F \rho(\tau) T^*} - 2||F||_\infty \beta_1 = \beta_1$$

By considering $\beta < \beta_2 := \frac{\beta_1}{2||F||_\infty} \inf_{[0,\tau]}(\frac{\rho}{\bar{\rho}} - 1)$, we conclude that:

$$d(x_\beta, t_\beta - \tau_\beta) \leq \rho(\tau'_\beta) d(x_\beta, t_\beta)$$

and using the fact that ρ is non decreasing and $\tau'_\beta \leq \tau_\beta$ this contradicts

$$d(x_\beta, t_\beta - \tau_\beta) > \rho(\tau_\beta) d(x_\beta, t_\beta)$$

Case 2: $t_\beta - \tau_\beta > T^*$. This implies that $\tau_\beta \in [0, \beta]$. By using the fact that d is semi-Lipschitz and

$$d(x_\beta, t_\beta - \tau_\beta) > \rho(\tau_\beta) d(x_\beta, t_\beta),$$

we get:

$$(\rho(\tau_\beta) - 1) d(x_\beta, T^*) < 2||F||_\infty \rho(\tau_\beta)\beta$$

By using the result that came from Gronwall's lemma, we finally get:

$$0 < \delta e^{-C_F \rho(\tau) T^*} < \frac{\rho(\tau_\beta)}{\rho(\tau_\beta) - 1} 2||F||_\infty \beta$$

which implies $\delta = 0$ for vanishing β.

Counter-example

Here we choose F such that $F = Id$ in $[0,1]$. Let $n_0 \in \mathbb{N}^*$ such that $\tau > \frac{2}{n_0}$.

Given two sets of drivers $(X_i)_{i\in\mathbb{Z}}$ and $(Y_i)_{i\in\mathbb{Z}}$ such that there exists $j \in \mathbb{Z}$ such that for $t \in [-\tau, 0]$:

$$\begin{cases} y_j^0(t) = j - 1 + \frac{n_0}{n_0+1} e^{n_0 t} < x_j^0(t) = j \\ y_{j+1}^0(t) = j + \frac{n_0}{n_0+1} e^{n_0 t} < x_{j+1}^0(t) = y_{j+1}^0(t) + \frac{1}{n_0+1} \end{cases}$$

By direct integration, we find:

$$(Y_j - X_j)(\tau) = \frac{n_0}{n_0+1}\tau - \frac{1}{n_0+1} - \frac{1 - e^{-n_0\tau}}{n_0+1} > 0$$

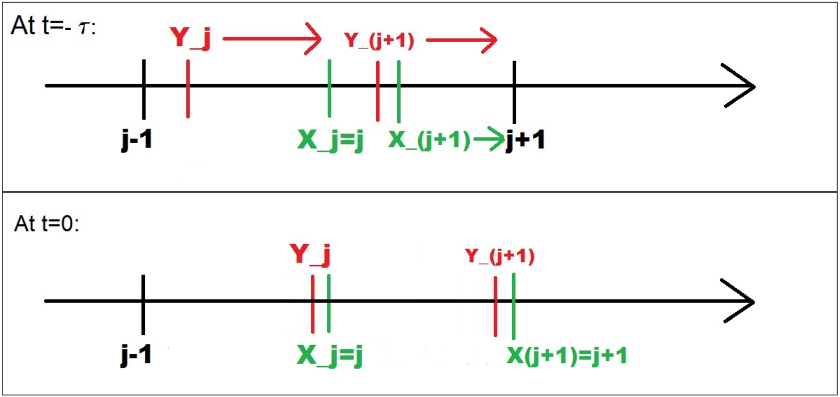

Figure : Initial dynamics for the counter-example.

Convergence proof (sketch)

1. Lipschitz estimates and barriers:
   - u^ε is $||F||_\infty -$Lipschitz in time.
   - Strict comparison principle: u^ε is $2L-$Lipschitz in space.
   - Barriers: $|u^\varepsilon(x,t) - u_0(x,t)| \leq (||F||_\infty + L)t$.

2. (Classical) use of relaxed semi-limits $\bar{u}$ and $\underline{u}$:
   If $\bar{u}$ is a subsolution of the HJ equation (and $\underline{u}$ is a supersolution) then by comparison principle:

$$\bar{u} \leq u^0 \leq \underline{u}$$

$$\Longrightarrow \text{ convergence}$$

$\bar{u}$ is a subsolution:
By contradiction, if not, there exist $(\bar{x}, \bar{t})$, $\varphi \in C^1_{x,t}$, $(r, \theta, \eta) \in (0, +\infty)^3$ such that:

$$\begin{cases} \bar{u} < \varphi \text{ in } B_{2r}(\bar{x}, \bar{t}) \setminus \{(\bar{x}, \bar{t})\} \\ \bar{u}(\bar{x}, \bar{t}) = \varphi(\bar{x}, \bar{t}) \\ \bar{u} \le \varphi - 2\eta \text{ in } B_{2r}(\bar{x}, \bar{t}) \setminus B_r(\bar{x}, \bar{t}) \end{cases}$$

and,

$$\partial_t \varphi(\bar{x}, \bar{t}) = 2\theta + F(\partial_x \varphi(\bar{x}, \bar{t}))$$

with $B_r(y, s) := (y - r, y + r) \times (s - r, s + r)$

We define:

$$M_\varepsilon := \max_{B_{2r}(\bar{x},\bar{t})} (u^\varepsilon - \varphi) := (u^\varepsilon - \varphi)(x_\varepsilon, t_\varepsilon)$$

For ε small enough:

$$(x_\varepsilon, t_\varepsilon) \in B_r(\bar{x}, \bar{t})$$

For $\varphi^\varepsilon := \varphi + M_\varepsilon$, we have (for ε small enough):

$$\begin{cases} \varphi^\varepsilon(x_\varepsilon t_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon) \\ \partial_t \varphi^\varepsilon(x,t) \geq F\left(\frac{\varphi^\varepsilon(x+\varepsilon, t-\varepsilon\tau) - \varphi^\varepsilon(x, t-\varepsilon\tau)}{\varepsilon}\right) + \frac{\theta}{2} \text{ in } B_r(\bar{x}, \bar{t}) \end{cases}$$

Use of localised strict comparison principle: $\varphi^\varepsilon > u^\varepsilon$ in $B_r(\bar{x}, \bar{t})$

Contradiction with $\varphi^\varepsilon(x_\varepsilon, t_\varepsilon) = u^\varepsilon(x_\varepsilon, t_\varepsilon)$

Thank you for your attention