June 2, 2016

A discrete isoperimetric inequality on lattices

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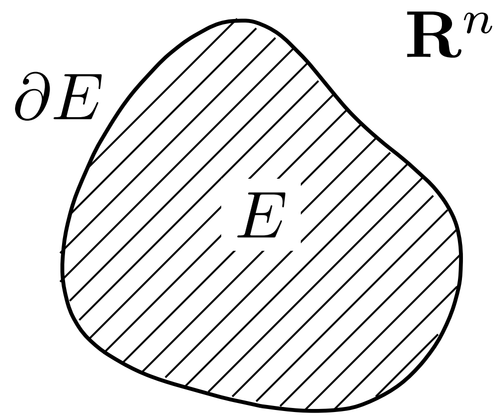
1 Introduction

Theorem (Classical isoperimetric inequality). For every bounded domain $E \subset \mathbf{R}^n$ with smooth boundary, we have

$$\frac{|\partial E|^n}{|E|^{n-1}} \geq \frac{|\partial B_1|^n}{|B_1|^{n-1}}.$$

The equality holds if and only if E is a ball.

▷ $B_r := \{|x| < r\}$ (ball), $|E| = \mathcal{L}^n(E)$, $|\partial E| = \mathcal{H}^{n-1}(\partial E)$.



Cabré's proof. [Cabré, '00 (Catalan), '08, '15 (preprint)].

A solution of a Poisson-Neumann problem on $E \subset \mathbf{R}^n$.

$\downarrow$ The method for proving the **ABP maximum principle**.

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Classical isoperimetric inequality

$$\frac{|\partial E|^n}{|E|^{n-1}} \geq \frac{|\partial B_1|^n}{|B_1|^{n-1}}.$$

Goal of this work:

A solution of a finite difference Poisson-Neumann problem on $\Omega \subset \mathbf{Z}^n$.

↓

Discrete isoperimetric inequality

$$\text{“} \quad \frac{|\partial \Omega|^n}{|\Omega|^{n-1}} \geq \frac{|\partial Q|^n}{|Q|^{n-1}}. \quad \text{”}$$

⋆ How should we define $|\Omega|$ and $|\partial\Omega|$? What is the optimal shape Q?

ABP maximum principle (Aleksandrov-Bakelman-Pucci)

A subsolution u of an elliptic problem

$$F(\nabla^2 u) = f(x) \quad \text{in } E$$

satisfies

$$\max_{\overline{E}} u \leq \max_{\partial E} u + C\|f\|_{L^n(\Gamma[u])},$$

where $C = C(n, \lambda, \mathrm{diam}(E))$. ([Cabré, '95] Improved ABP.)

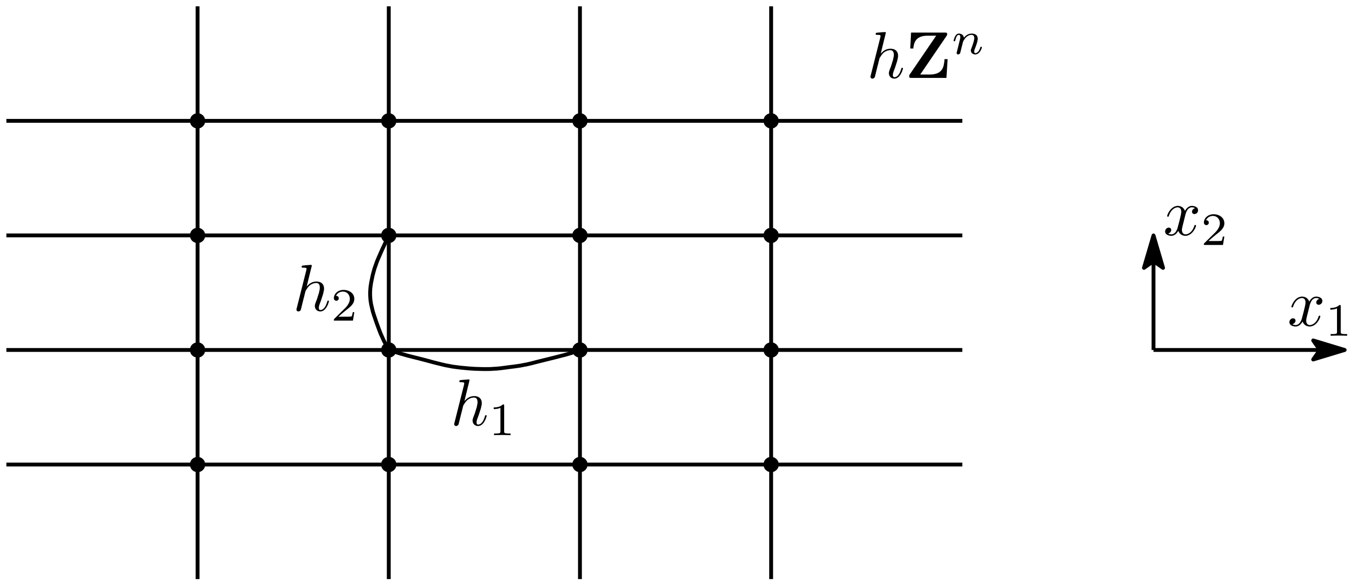
Discrete version.

- [Kuo-Trudinger, '90, '92, '93, '96, '98, '00 etc.] ABP etc.
- [H., '16] Harnack.

A result

Definition (Lattice). • Let $h_1, \ldots, h_n > 0$.

$$h\mathbf{Z}^n := \{(h_1 x_1, \ldots, h_n x_n) \in \mathbf{R}^n \mid (x_1, \ldots, x_n) \in \mathbf{Z}^n\}.$$



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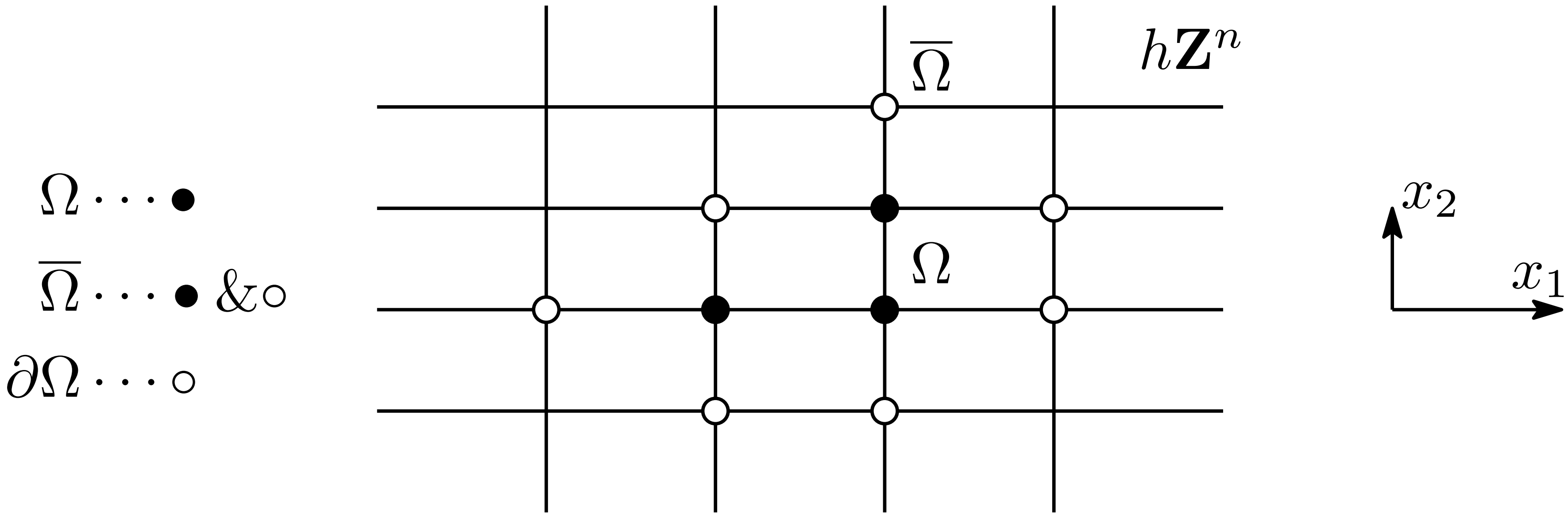
• Let $\Omega \subset h\mathbf{Z}^n$ and $\{e_i\}_{i=1}^n$ be the standard basis in $\mathbf{R}^n$.

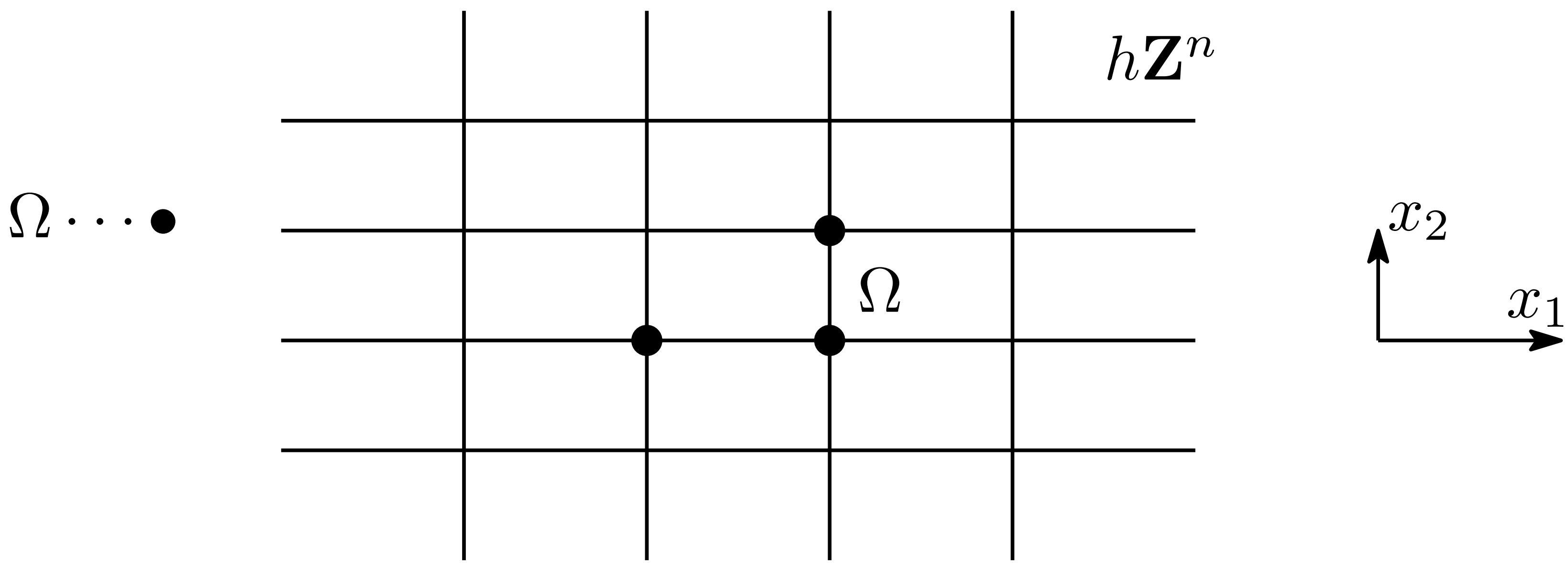
$$\overline{\Omega} := \{x + \sigma h_i e_i \mid x \in \Omega,\ i \in \{1, \ldots, n\},\ \sigma \in \{-1, 0, 1\}\},$$

$$\partial\Omega := \overline{\Omega} \setminus \Omega.$$



Definition (Volume and Perimeter). Let $\Omega \subset h\mathbf{Z}^n$.

$$E[\Omega] := \bigcup_{(x_1,\ldots,x_n)\in\Omega} \left[x_1 - \frac{h_1}{2}, x_1 + \frac{h_1}{2}\right] \times \cdots \times \left[x_n - \frac{h_n}{2}, x_n + \frac{h_n}{2}\right].$$



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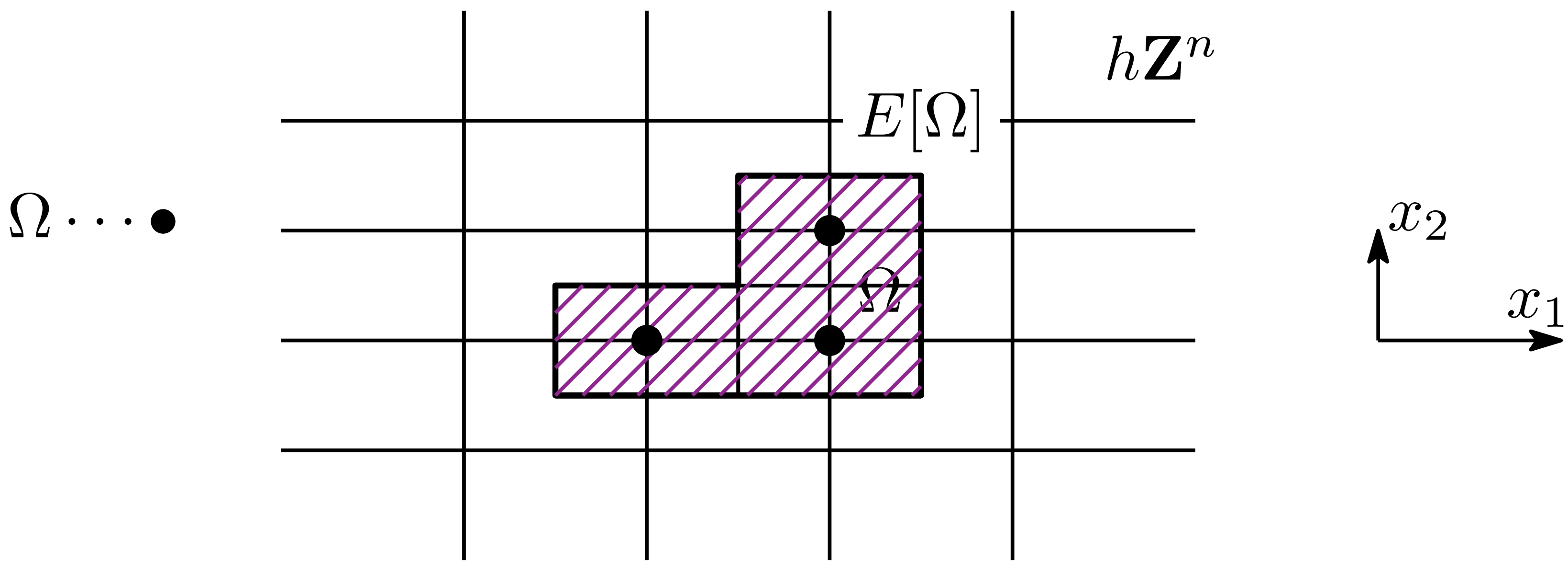

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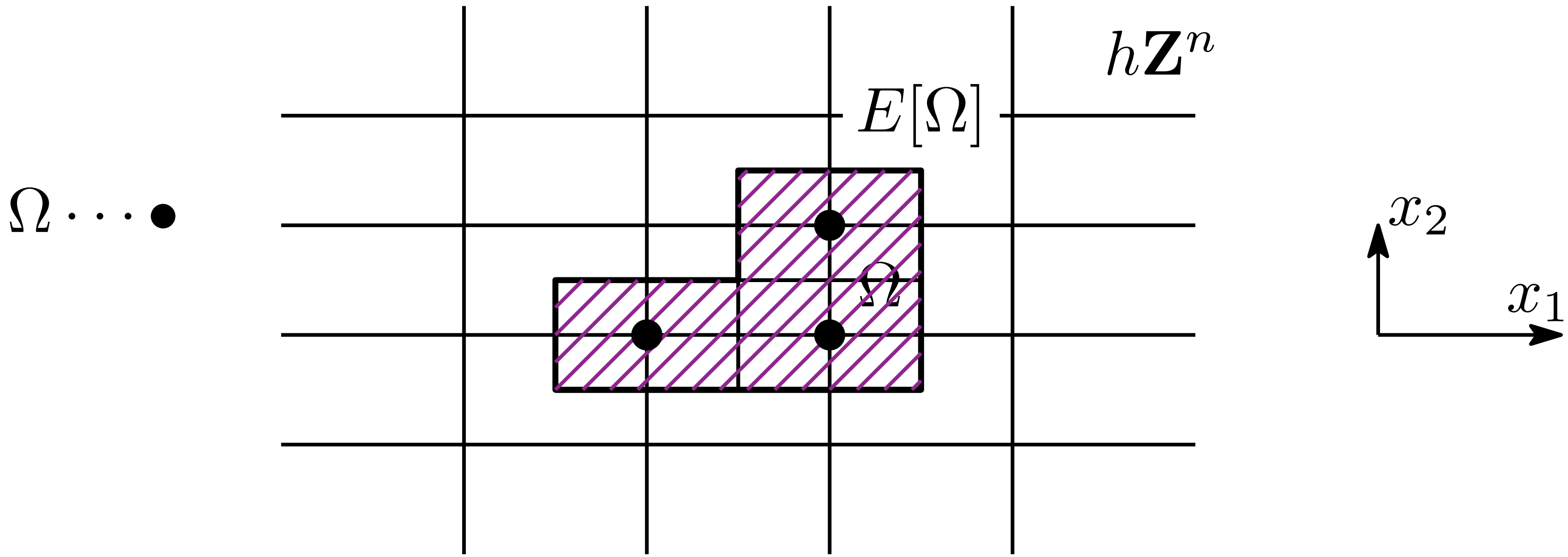
$$\mathrm{Vol}(\Omega) := |E[\Omega]|, \quad \mathrm{Per}(\Omega) := |\partial E[\Omega]|.$$



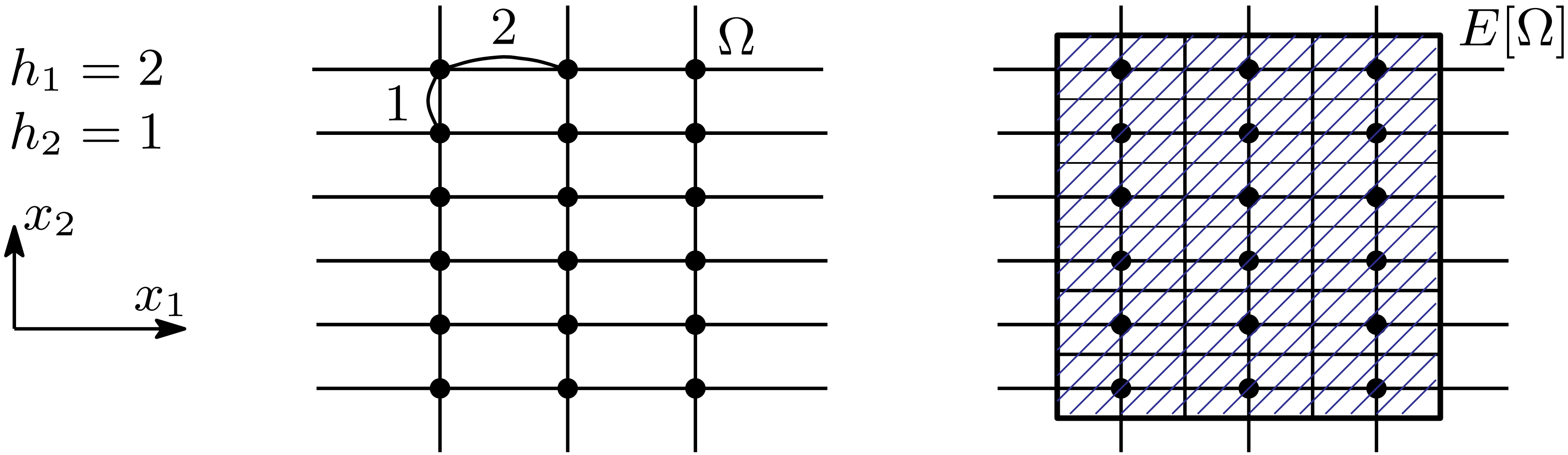
▷ $Q_r := (-r, r)^n \subset \mathbf{R}^n$ (cube).

Theorem (Discrete isoperimetric inequality). For every bounded $\Omega \subset h\mathbf{Z}^n$, we have

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geqq \frac{|\partial Q_1|^n}{|Q_1|^{n-1}}.$$

The equality holds if and only if $E[\Omega]$ is a cube.

Example. The equality case:

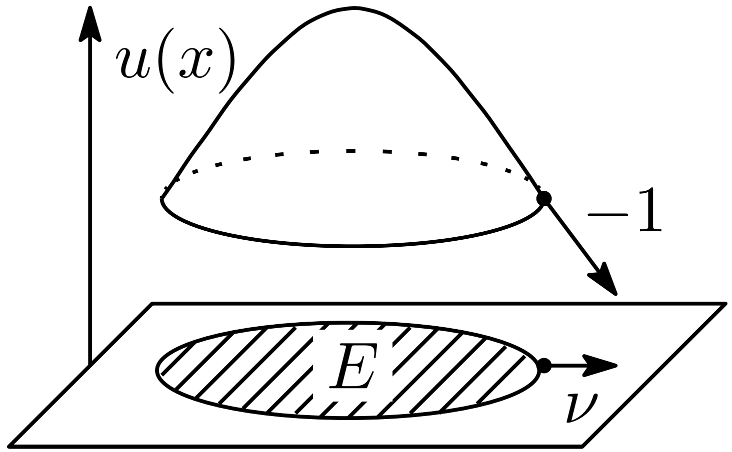


2 Cabré's idea ([Cabré, '00 (Catalan), '08])

Proof of the classical isop.

Let $u \in C^2(E) \cap C(\overline{E})$ be a solution of Poisson-Neumann problem:

$$(\mathrm{PN}) \begin{cases} -\Delta u = \dfrac{|\partial E|}{|E|} & \text{in } E, \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \partial E. \end{cases}$$



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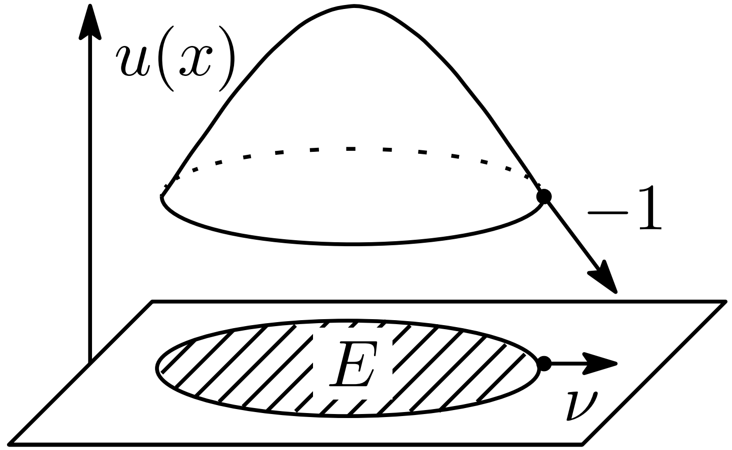

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Remark.

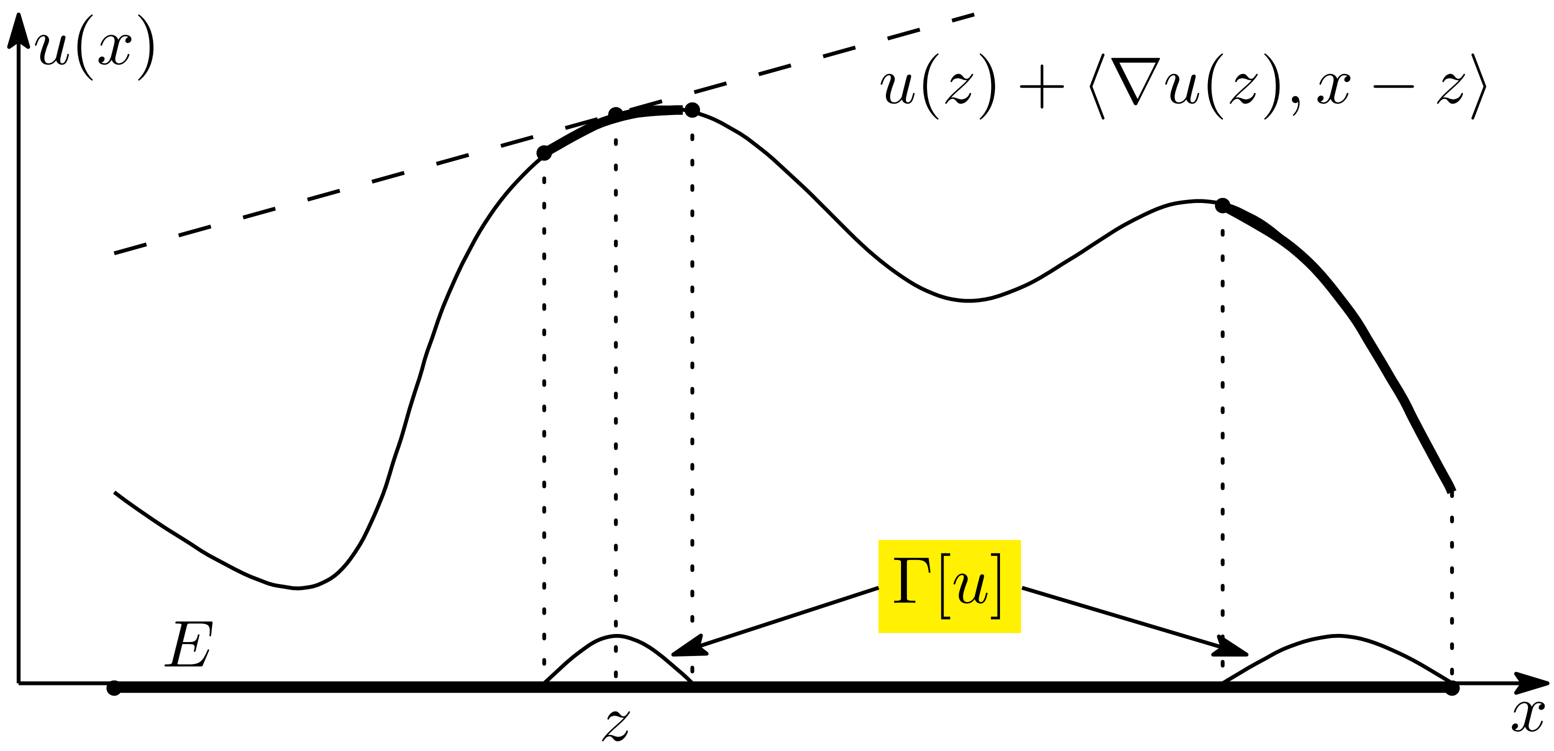
- (PN) satisfies the compatibility condition:

$$\int_E f\,dx + \int_{\partial E} g\,dS = 0,$$

which is needed to solve $-\Delta u = f$ in E, $\dfrac{\partial u}{\partial \nu} = g$ on ∂E.

Define the **upper contact set** $\Gamma[u](\subset E)$ of u as

$$\Gamma[u] := \{z \in E \mid u(x) \leqq \underbrace{u(z) + \langle \nabla u(z), x - z \rangle}_{\text{Tangent plane at } z},\ \forall x \in \overline{E}\}.$$



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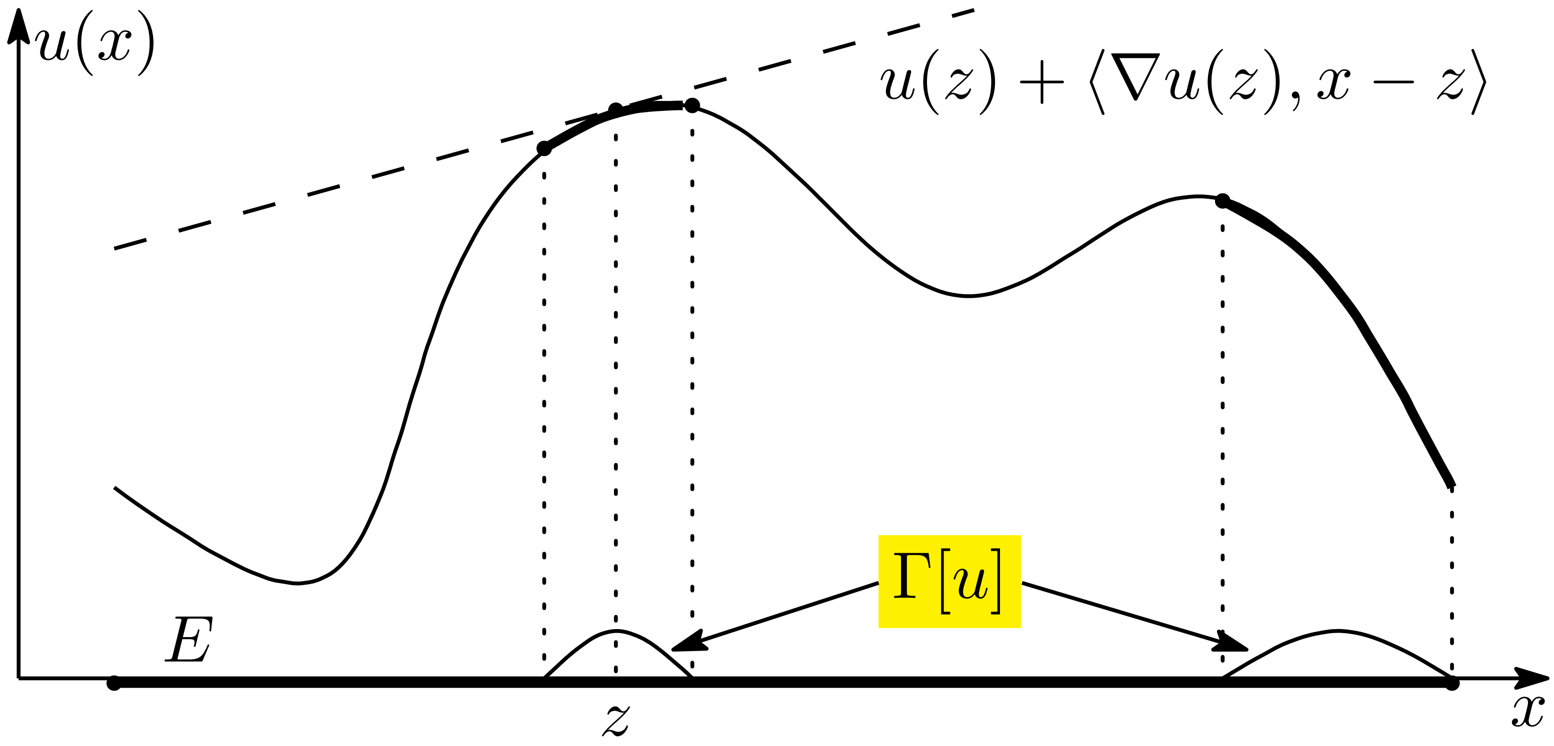
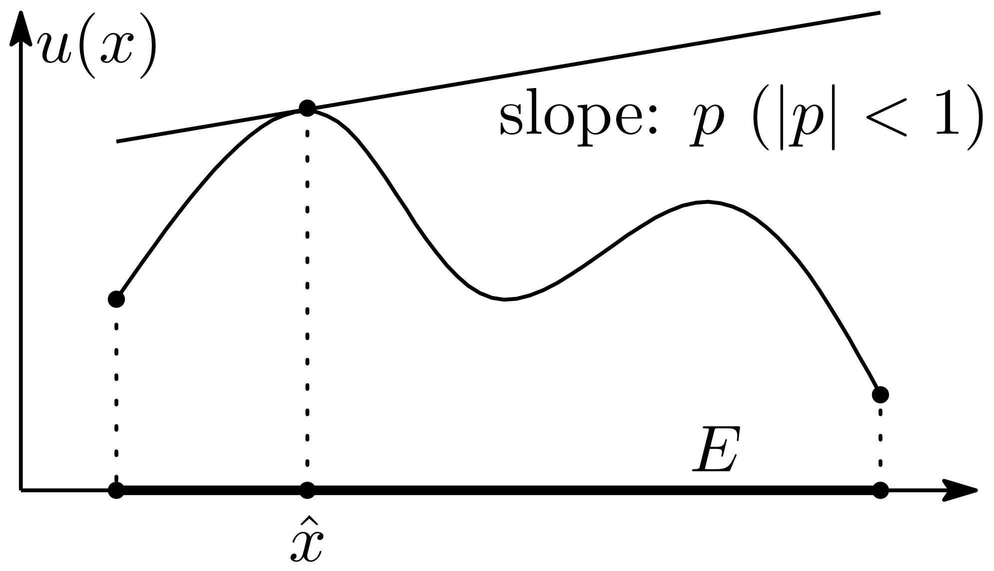


$$\nabla^2 u \leqq O \text{ on } \Gamma[u].$$

$\triangleright$ $\Gamma[u] := \{z \in E \mid u(x) \leq u(z) + \langle \nabla u(z), x - z \rangle,\ \forall x \in \overline{E}\}$.

Claim. $B_1 \subset \nabla u(\Gamma[u])$.



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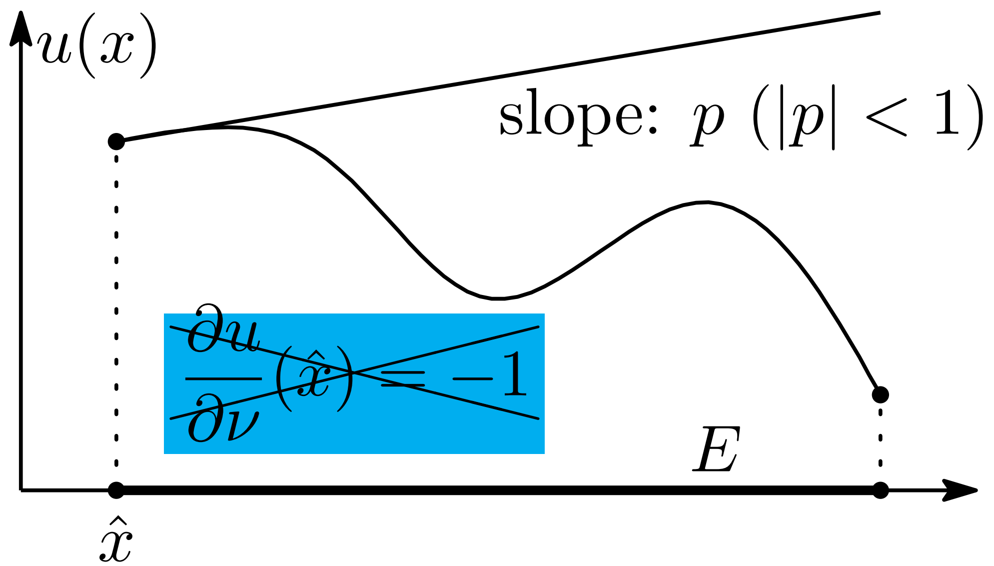

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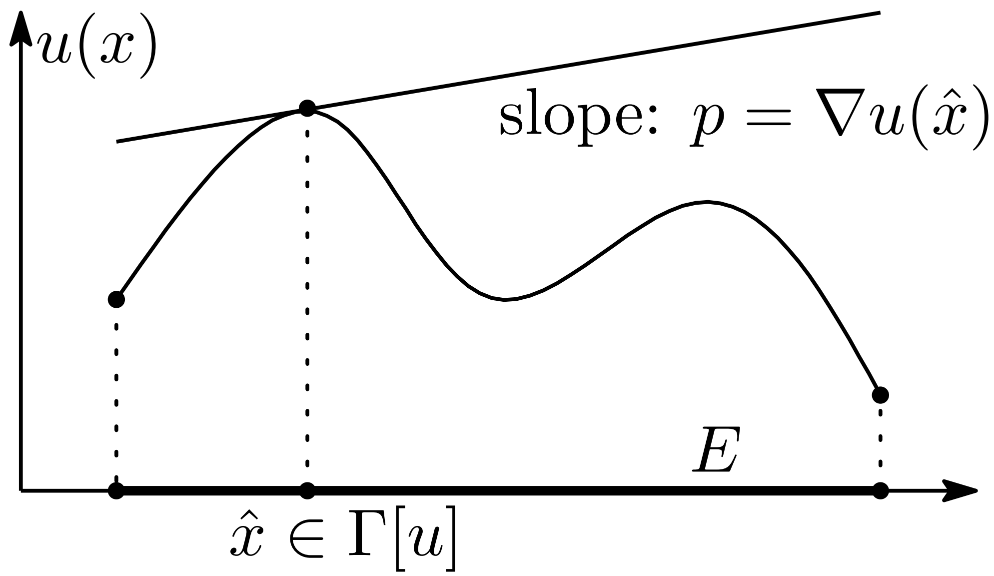

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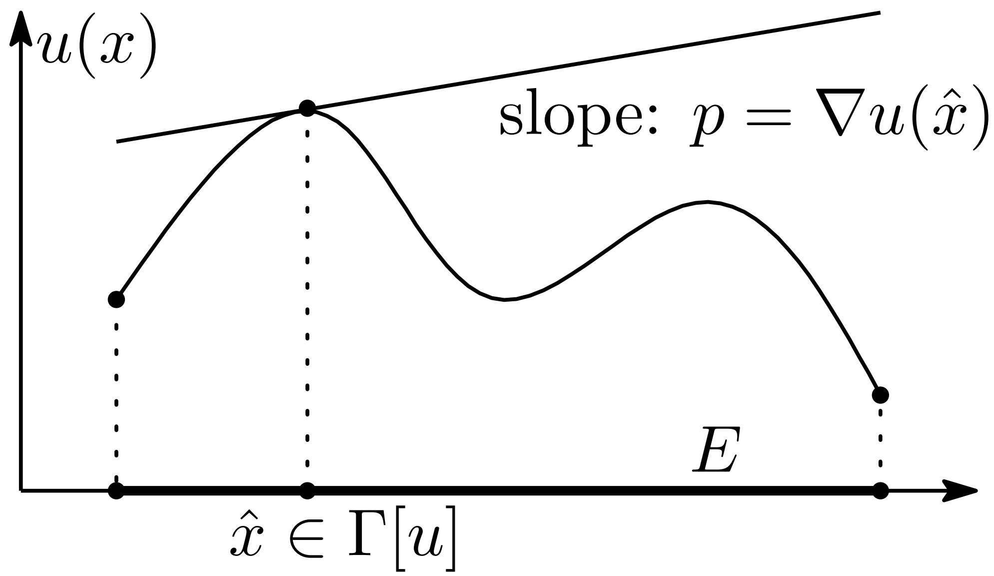

Remark.

In general, $\boxed{B_d \subset \nabla u(\Gamma[u])}$ with $d = \dfrac{\max_{\overline{E}} u - \max_{\partial E} u}{\operatorname{diam}(E)} \implies$ ABP.

By the claim and the change of variables (the area formula)

$$|B_1| \overset{\text{Claim}}{\leqq} |\nabla u(\Gamma[u])| = \int_{\nabla u(\Gamma[u])} dp \overset{\text{CV}}{\leqq} \int_{\Gamma[u]} |\det \nabla^2 u| dx. \qquad \text{(Det)}$$

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Denote by $-e_j$ ($e_j \geqq 0$) the eigenvalues of $\nabla^2 u \, (\leqq O)$ on $\Gamma[u]$. Then

$$|\det \nabla^2 u| = e_1 \times \cdots \times e_n \overset{\text{AMGM}}{\leqq} \left(\frac{e_1 + \cdots + e_n}{n} \right)^n = \left(\frac{-\Delta u}{n} \right)^n = \left(\frac{|\partial E|}{n|E|} \right)^n.$$

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Substituting this for (Det),

$$|B_1| \leqq \int_{\Gamma[u]} \left(\frac{|\partial E|}{n|E|}\right)^n dx \leqq |E| \left(\frac{|\partial E|}{n|E|}\right)^n = \frac{|\partial E|^n}{n^n |E|^{n-1}}.$$

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Finally, applying $n = |\partial B_1|/|B_1|$ yields

$$\frac{|\partial E|^n}{|E|^{n-1}} \geqq n^n |B_1| = \left(\frac{|\partial B_1|}{|B_1|}\right)^n |B_1| = \frac{|\partial B_1|^n}{|B_1|^{n-1}}. \quad \square$$

By the claim and the change of variables (the area formula)

This proof works even if u is only a subsol.: $-\Delta u \leq \dfrac{|\partial E|}{|E|}$

D

$$|\det \nabla^2 u| = e_1 \times \cdots \times e_n \overset{\text{AMGM}}{\leq} \left(\frac{e_1 + \cdots + e_n}{n} \right)^n = \left(\frac{-\Delta u}{n} \right)^n = \left(\frac{|\partial E|}{n|E|} \right)^n.$$

$\leq$

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Variations

- [Cabré-RosOton, '13, JDE] $m(E) = \int_E x^A$, $P(E) = \int_{\partial E} x^A$ (weighted).

$$\begin{cases} -\mathrm{div}(x^A \nabla u) = \dfrac{P(E)}{m(E)} x^A & \text{in } E, \\ \dfrac{\partial u}{\partial \nu} = -1 & \text{on } \partial E. \end{cases}$$

- [Cabré-RosOton-Serra, preprint] $P_H(E) = \int_{\partial E} H(\nu(x))$ (anisotropic).

$$\begin{cases} -\Delta u = \dfrac{P_H(E)}{|E|} & \text{in } E, \\ \dfrac{\partial u}{\partial \nu} = -H(\nu) & \text{on } \partial E. \end{cases}$$

- [Trudinger, '94] Isoperimetric inequalities via Monge-Ampère equations.

3 Discretization

Issues.

- Interpretation of ∇u on lattices and its estimate.

- Solvability of a finite difference Poisson-Neumann problem.

- The case of equality.

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- Interpretation of ∇u on lattices and its estimate.

Use the superdifferential $\partial^+ u$. (Step 1)

- Solvability of a finite difference Poisson-Neumann problem.

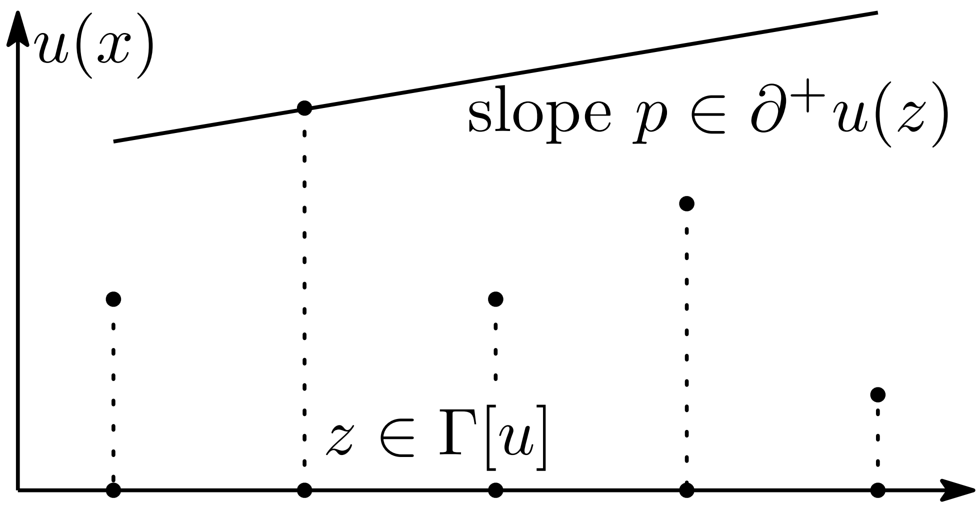
$\exists$ subsolution $\Longleftarrow$ Definition of $\mathrm{Vol}(\cdot)$, $\mathrm{Per}(\cdot)$. (Step 2)

- The case of equality.

Observe $\partial^+ u$ adjacent to one another. (Step 3)

Step 1: Superdifferentials

$$\partial^+ u(z) := \{p \in \mathbf{R}^n \mid u(x) \leqq u(z) + \langle p, x - z\rangle,\ \forall x \in \overline{\Omega}\}.$$



$$\begin{aligned}\Gamma[u] :=& \{z \in \Omega \mid \exists p \in \mathbf{R}^n \text{ s.t. } u(x) \leqq u(z) + \langle p, x - z\rangle,\ \forall x \in \overline{\Omega}\} \\ =& \{z \in \Omega \mid \partial^+ u(z) \neq \emptyset\}.\end{aligned}$$

cf. [Kuo-Trudinger, '90 etc.]

Set

$$\delta_i^{\pm} u(x) := \frac{u(x \pm h_i e_i) - u(x)}{\pm h_i}, \quad \delta_i^2 u(x) = \frac{\delta_i^+ u(x) - \delta_i^- u(x)}{h_i}, \quad \Delta' := \sum_{i=1}^{m} \delta_j^2.$$

Fact.

$$\partial^+ u(z) \subset [\delta_1^+ u(z), \delta_1^- u(z)] \times \cdots \times [\delta_n^+ u(z), \delta_n^- u(z)] \quad (\forall z \in \Gamma[u]).$$

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Fact.

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Taking the Lebesgue measure ($h^n := h_1 \times \cdots \times h_n$),

$$|\partial^+ u(z)| \leqq h^n(-\delta_1^2 u(z)) \times \cdots \times (-\delta_n^2 u(z)) \leqq h^n \left(\frac{-\Delta' u(z)}{n}\right)^n,$$

a substitute for the area formula: $|\nabla u(\Gamma[u])| \leqq \int_{\Gamma[u]} |\det \nabla^2 u| dx.$

Step 2: Solvablity of the discrete Poisson-Neumann problem

Proposition. Let $\Omega \subset h\mathbf{Z}^n$ be bounded. Then there exists $u : \overline{\Omega} \to \mathbf{R}$ such that

$$\begin{cases} \forall x \in \Omega, & -\Delta' u(x) \leq \dfrac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)}, \\ \forall x \in \partial\Omega, \exists y_x \in \overline{\{x\}} \cap \Omega \ \text{ s.t.} & \dfrac{u(x) - u(y_x)}{h_i} = -1, \end{cases}$$

where i is the direction which is parallel to the edge xy_x.

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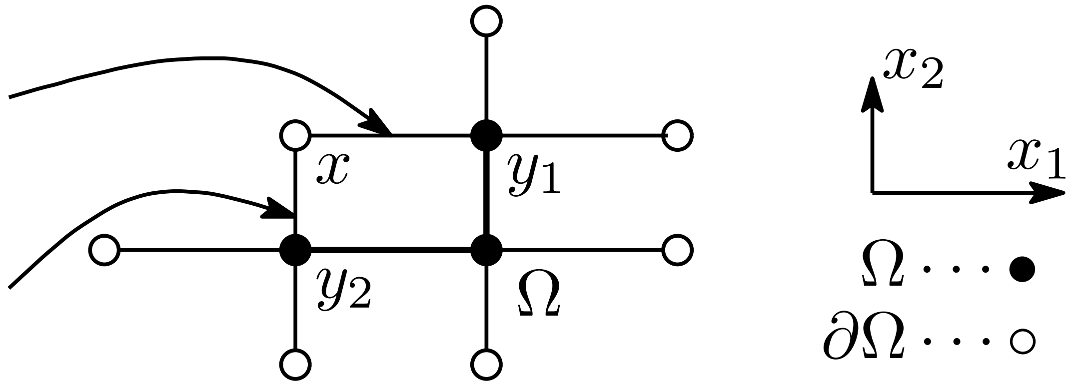
$$\frac{u(x) - u(y_1)}{h_1} = -1$$

or $$\frac{u(x) - u(y_2)}{h_2} = -1$$



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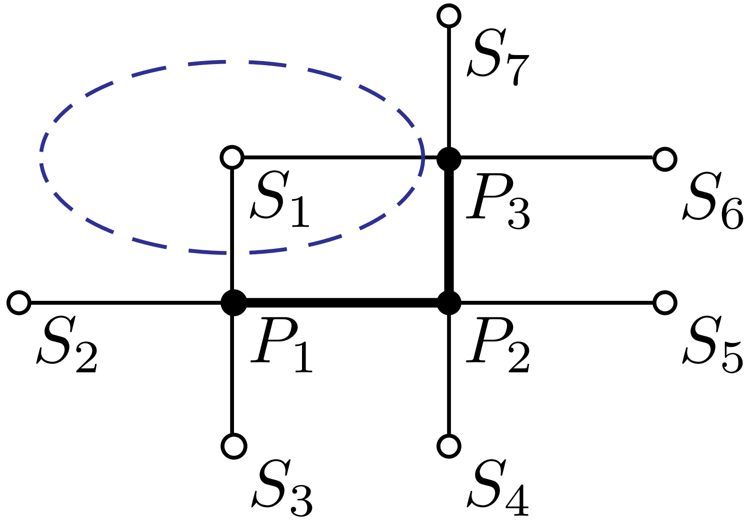
This u satisfies

$$Q_1 \subset \bigcup_{z \in \Gamma[u]} \partial^+ u(z), \qquad Q_1 = (-1,1)^n.$$

(Classical case: $B_1 \subset \nabla u(\Gamma[u])$.)

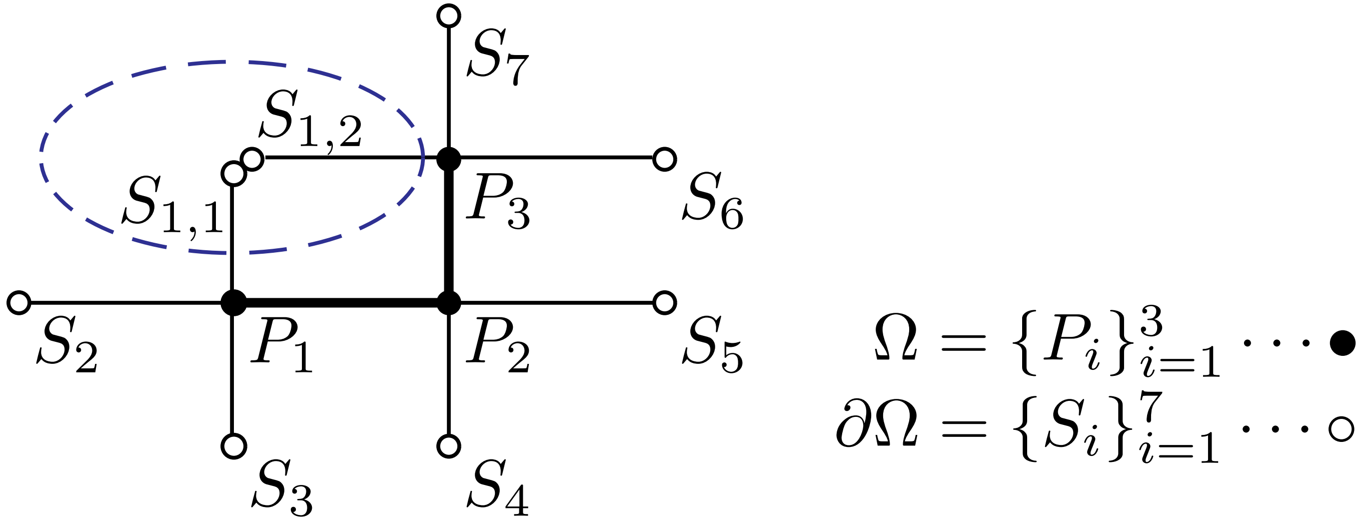


$\Omega = \{P_i\}_{i=1}^{3} \cdots \bullet$

$\partial\Omega = \{S_i\}_{i=1}^{7} \cdots \circ$

Proof. 1. We regard $S_1 \in \partial\Omega$ as two different points $S_{1,1}$ and $S_{1,2}$, and solve the corresponding system of linear equations.

$$\left(-\Delta' u(P_*) = \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} \text{ in } \Omega, \qquad \frac{u(S_*) - u(P_*)}{h_i} = -1 \text{ on } \partial\Omega. \right)$$



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$\star$ The number of equations is equal to that of the unknowns.

2. We define $u : \overline{\Omega} \to \mathbf{R}$ by

$$\boxed{u(S_1) := \max\{u(S_{1,1}),\ u(S_{1,2})\}}.$$

Then u is a subsolution and satisfies the Neumann condition. $\square$

The equation is written as

$$L\vec{a} = \vec{b}$$

with L: symmetric & $\mathrm{Ker} L = \mathbf{R}\vec{1}$.

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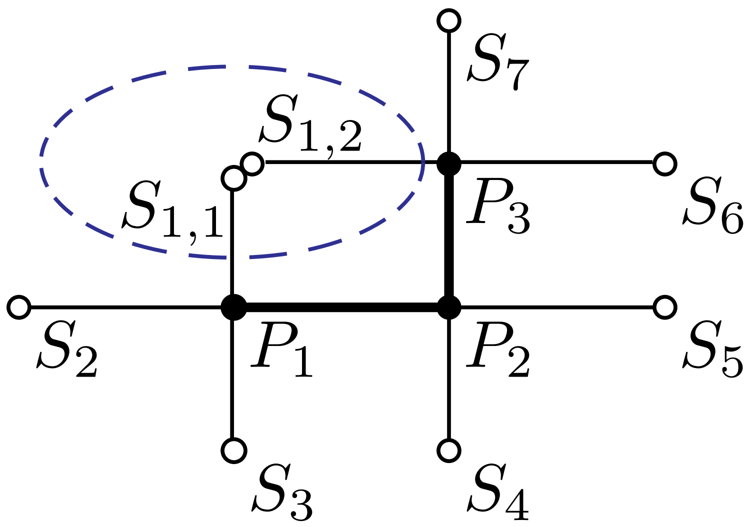

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For the above example

$$\vec{a} = {}^t[u(P_1)\ u(P_2)\ u(P_3) \parallel u(S_{1,1})\ u(S_{1,2}) \mid u(S_2)\ u(S_3)\ \ldots\ u(S_7)],$$

$$\vec{b} = {}^t\left[\frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)}\ \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)}\ \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} \,\Big\|\, \frac{-1}{h_2}\ \frac{-1}{h_1} \,\Big|\, \frac{-1}{h_1}\ \frac{-1}{h_2}\ \ldots\ \frac{-1}{h_2}\right].$$

▷ $L\vec{a} = \vec{b}$.

Remark. Another representation of Vol(·) & Per(·):

$$\mathrm{Vol}(\Omega) = |E[\Omega]| = h^n \times (\#\Omega), \quad \mathrm{Per}(\Omega) = |\partial E[\Omega]| = \sum_{i=1}^{n} \frac{h^n}{h_i} \times \omega_i.$$

with ω_i: # (edges in the x_i-direction connecting Ω & $\partial\Omega$).

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$$\Longrightarrow \vec{b} \in \mathrm{Im}L.$$

$$(\mathrm{Im}L = (\mathbf{R}\vec{1})^{\perp},\ \langle \vec{b}, \vec{1} \rangle = \sum \vec{b} = 0)$$

Step 3: The case of equality

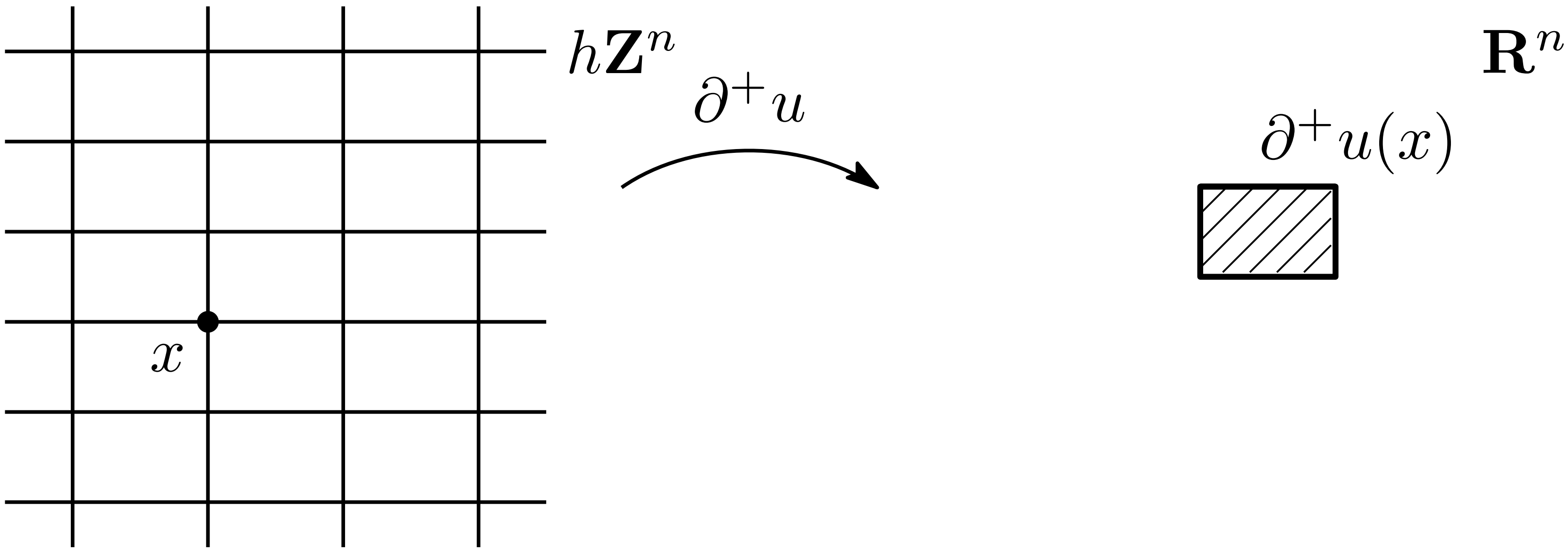
The equality is attained if $\Omega = \Gamma[u]$ and

$$\partial^+ u(x) = [\delta_1^+ u(x), \delta_1^- u(x)] \times \cdots \times [\delta_n^+ u(x), \delta_n^- u(x)] \ (\forall x \in \Omega), \quad (*1)$$

$$\delta_1^2 u(x) = \cdots = \delta_n^2 u(x) \ (\forall x \in \Omega), \quad (*2)$$

$$\overline{Q_1} = \bigcup_{x \in \Omega} \partial^+ u(x). \quad (*3)$$

$(*1)$ $\Longrightarrow$ A superdiff. $\partial^+ u(x)$ is a hyperrectangle for every $x \in \Omega$.



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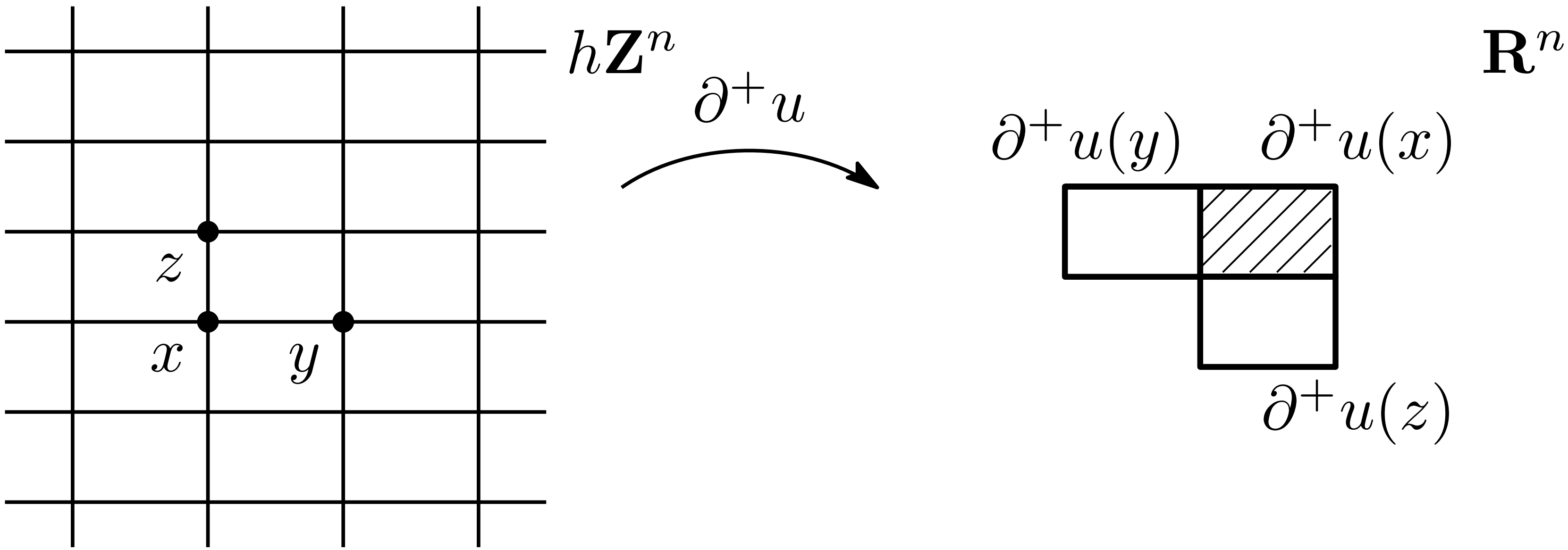
(*2) $\Longrightarrow$ Superdiff. at pts adjacent to x are congruent with $\partial^+ u(x)$.



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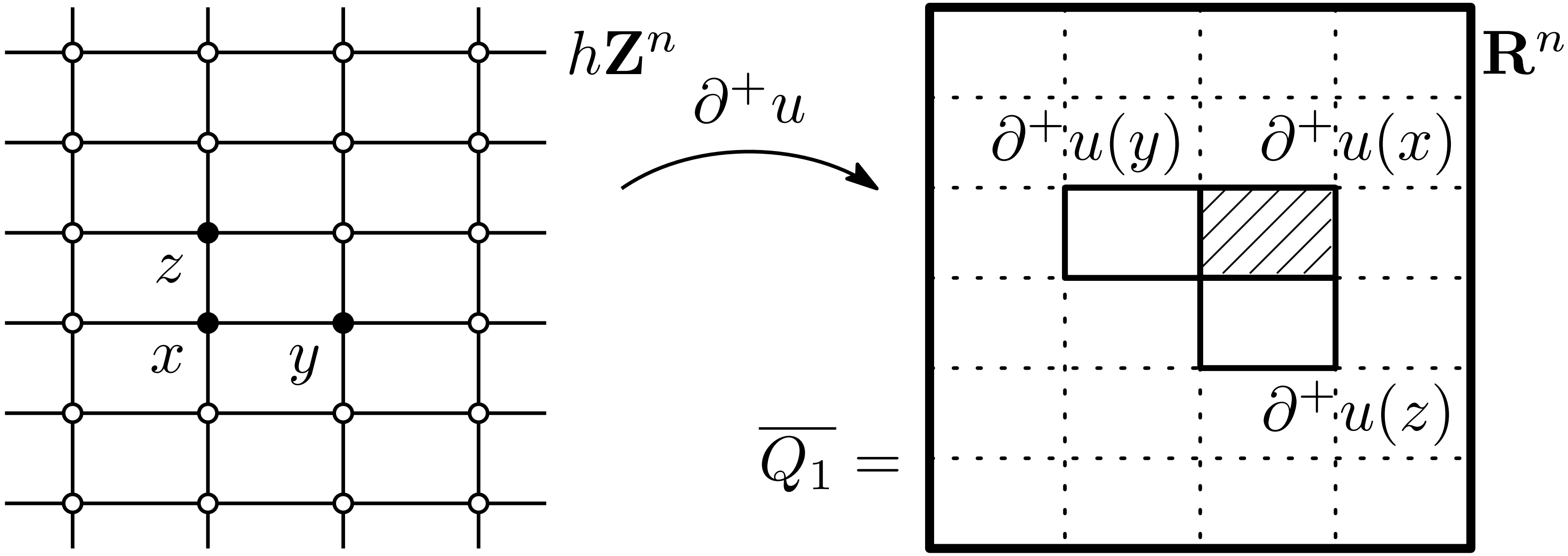
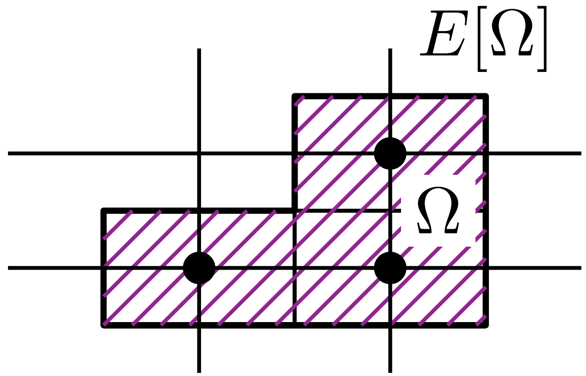
(∗3) $\Longrightarrow$ Ω must be a cube since the union of $\partial^+ u$ is Q_1. $\square$



4 Summary

- We derived a discrete isoperimetric inequality:

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geq \frac{|\partial Q_1|^n}{|Q_1|^{n-1}}$$



for a subset Ω of lattices via the ABP method.
Here $\mathrm{Per}(\Omega) = |E[\Omega]|$, $\mathrm{Vol}(\Omega) = |\partial E[\Omega]|$.

- An optimal shape is a cube.

- To derive the inequality, we proved an existence of (sub)solution to a discrete Poisson-Neumann problem.

Reference. [1] N. H., *A discrete isoperimetric inequality on lattices*, Discrete Comput. Geom. 52 (2014), no. 2, 221–239.

Extension

We must address the following issues in a consistent way:

(1) definitions of a volume and a perimeter;

(2) a definition of a discrete Laplace operator Δ';

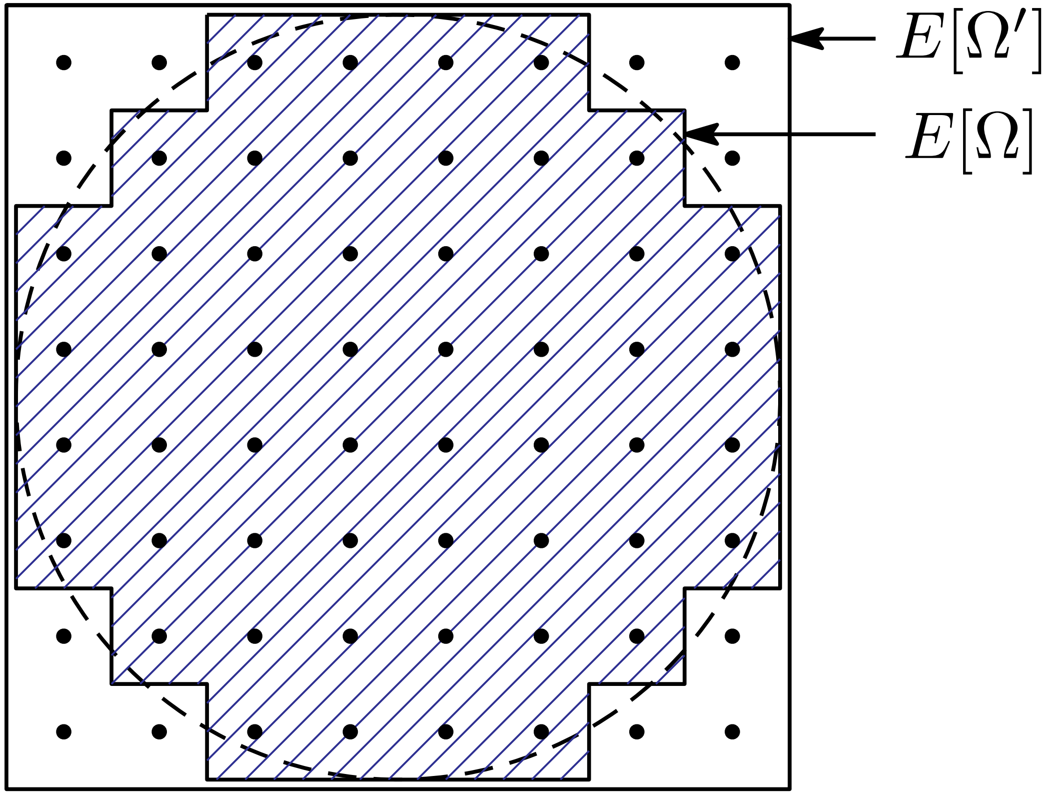
(3) an optimal estimate for $\mathcal{L}^n(\partial^+ u(z))$ by $-\Delta' u(z)$ for $z \in \Gamma[u]$;

(4) solvability of the Poisson-Neumann problem:

$$-\Delta' u = \frac{\mathrm{Per}(\Omega)}{\mathrm{Vol}(\Omega)} \quad \text{in } \Omega, \qquad \text{“}\frac{\partial u}{\partial \nu} = -1\text{”} \quad \text{on } \partial\Omega.$$

Remark. **"Round-shaped" Ω is not optimal.**

• $n = 2$. Consider a minimal rectangle Ω' covering a round-shaped Ω.



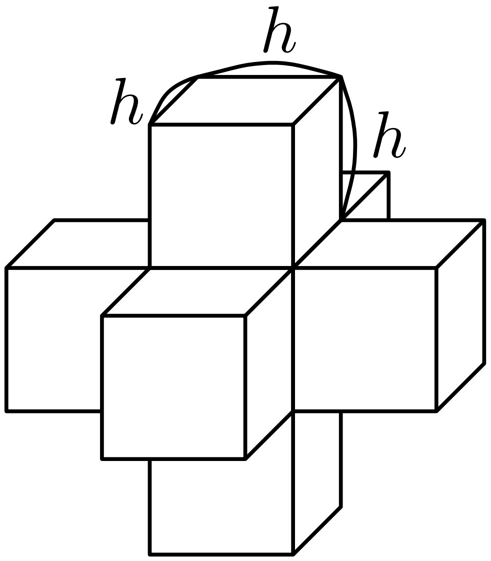
Since $\mathrm{Per}(\Omega) = \mathrm{Per}(\Omega')$ & $\mathrm{Vol}(\Omega) < \mathrm{Vol}(\Omega')$, we have

$$\frac{\mathrm{Per}(\Omega)^2}{\mathrm{Vol}(\Omega)} > \frac{\mathrm{Per}(\Omega')^2}{\mathrm{Vol}(\Omega')}.$$

This argument does not work if $n \geq 3$.

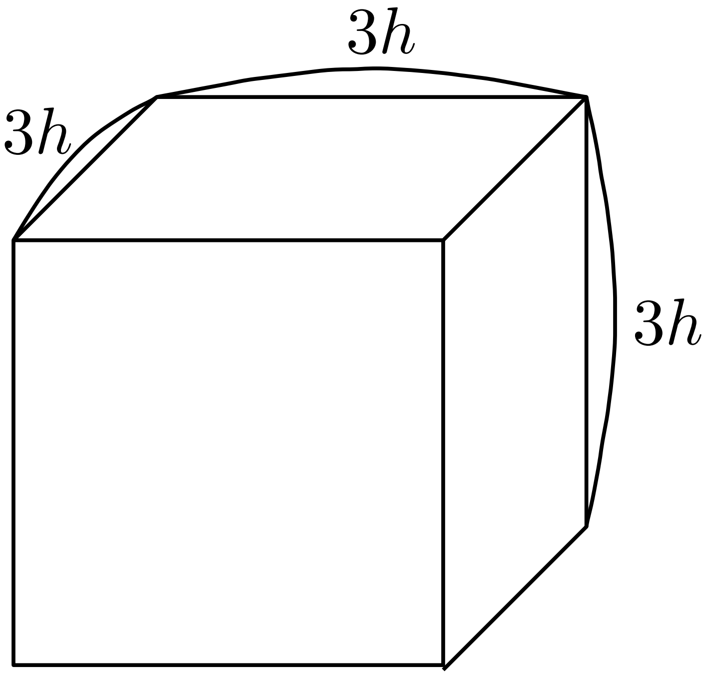
• <u>$n = 3$.</u> Let $h_1 = h_2 = h_3 = h > 0$.

$E[\Omega] =$



$\mathrm{Per}(\Omega) = 30h^2$

$E[\Omega'] =$



$\mathrm{Per}(\Omega') = 54h^2$

$\mathrm{Per}(\cdot)$ may increase.

$\triangleright$ (*) $|\partial^+ u(z)| \leqq h^n \left(\frac{-\Delta' u(z)}{n}\right)^n$

$$|Q_1| \overset{\text{Claim}}{\leqq} \left| \bigcup_{z \in \Gamma[u]} \partial^+ u(z) \right| \leqq \sum_{z \in \Gamma[u]} |\partial^+ u(z)|.$$

Using (*) and the fact that u is a subsolution,

$$|Q_1| \leqq \sum_{z \in \Gamma[u]} h^n \left(\frac{\mathrm{Per}(\Omega)}{n\mathrm{Vol}(\Omega)}\right)^n \leqq \underbrace{(\#\Omega) \times h^n}_{=\mathrm{Vol}(\Omega)} \left(\frac{\mathrm{Per}(\Omega)}{n\mathrm{Vol}(\Omega)}\right)^n = \frac{\mathrm{Per}(\Omega)^n}{n^n \mathrm{Vol}(\Omega)^{n-1}}.$$

Finally, applying $n = |\partial Q_1| / |Q_1|$,

$$\frac{\mathrm{Per}(\Omega)^n}{\mathrm{Vol}(\Omega)^{n-1}} \geqq n^n |Q_1| = \left(\frac{|\partial Q_1|}{|Q_1|}\right)^n |Q_1| = \frac{|\partial Q_1|^n}{|Q_1|^{n-1}}.$$