Discontinuous solutions of Hamilton-Jacobi-Bellman equations on networks

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joint work with J. GRABER, H. ZIDANI

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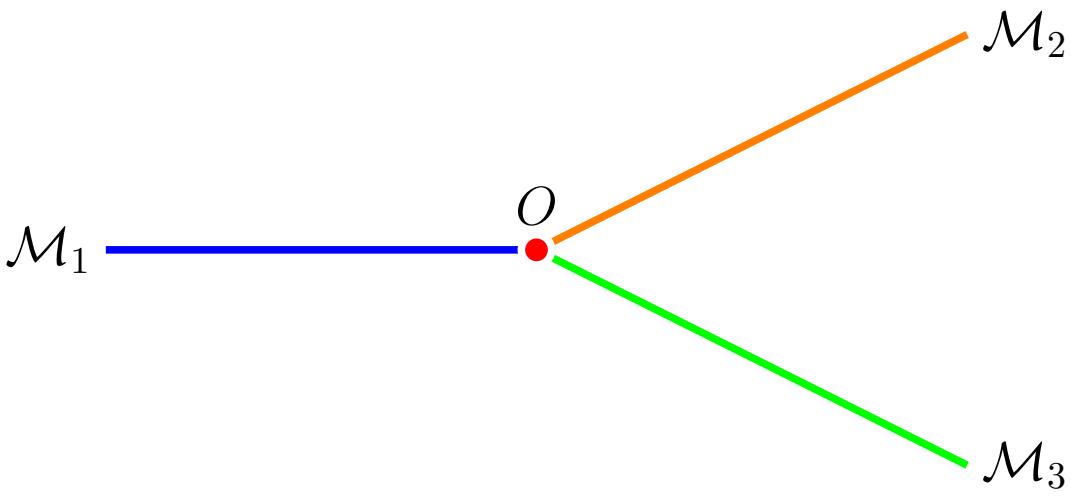
Basic model of a network



A set $\mathcal{K}$ is called a (1-dimensional) network if:

$$\mathcal{K} = \mathcal{O} \cup \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_p,$$

where :

- $\mathcal{O} = \{o\}$ is the junction
- $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are the branches

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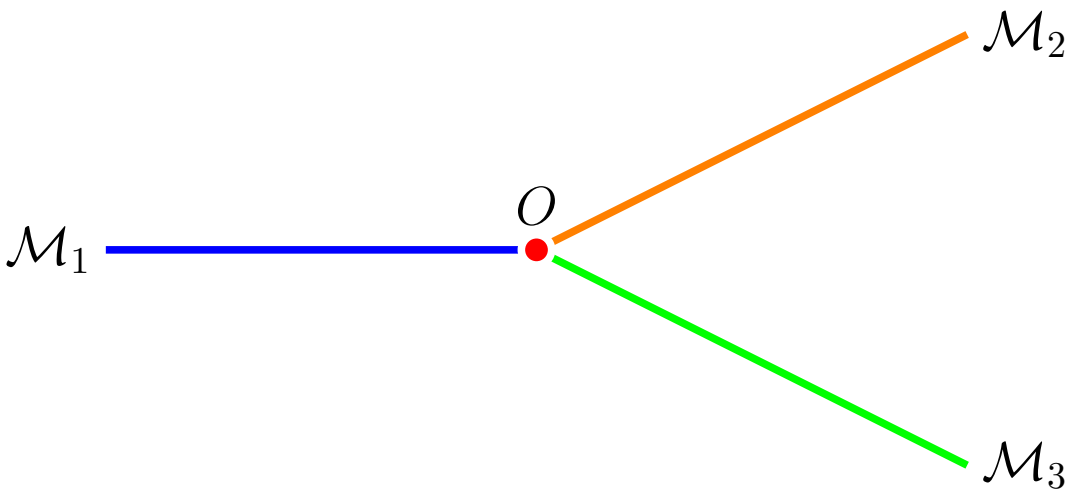

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- $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are the branches

Simplification: Each $\mathcal{M}_i = (0, +\infty)e_i$, with $|e_i| = 1$.

Problem of concern

We consider on each branch a HJ equation

$$\text{(HJ)} \qquad \begin{cases} -\partial_t u(t,x) + H_i(x, \nabla_x u(t,x)) = 0, & \forall (t,x) \in (0,T) \times \mathcal{M}_i, \\ u(T,x) = g_i(x), & \forall x \in \mathcal{M}_i. \end{cases}$$

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Under suitable assumptions,

$$\vartheta(t,x) := \inf \left\{ \int_t^T L(y(s), \dot{y}(s)) ds + g(y(T)) \,\middle|\, y \in \mathbb{S}^T(t,x) \right\}$$

solves (HJ), where $\mathbb{S}^T(t,x)$ is an appropriate subset of $AC([0,T];\mathbb{R}^N)$.

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Goal

Find junction conditions in order to make ϑ the unique solution of (HJ)

Bellman case

We focus on the Bellman case

$$H_i(x,p) = \sup_{v \in F_i(x)} \{-\langle v e_i, p\rangle - L_i(x,v)\}$$

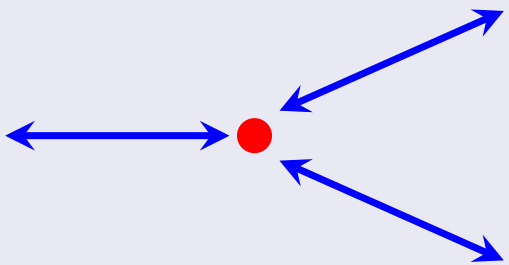
Optimal control point of view

On each branch there is a dynamical system

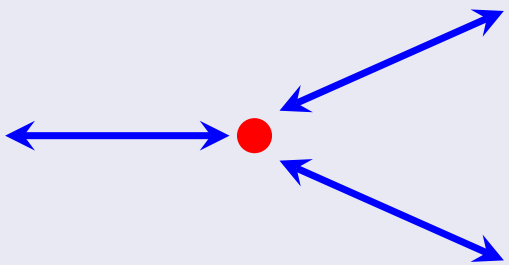
$$\dot{y}(s) \in F_i(y(s))e_i \quad \text{a.e. on } (t_1,t_2)$$

and an integral cost

$$y(s) \in \mathcal{M}_i \text{ on } (t_1,t_2) \quad \Longrightarrow \quad \int_{t_1}^{t_2} L_i(y(s),\dot{y}(s))ds$$



Achdou et al '12, Camilli et al '13, Imbert et al '13, Achdou et al '15, ...
Higher dimension: Barles et al '13 & '15, Oudet (preprint)



Achdou et al '12, Camilli et al '13, Imbert et al '13, Achdou et al '15, ...
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Essential assumption: $[-\delta, \delta] \subseteq F_i(o)$

- PDE: Coercivity of the Hamiltonians.
- OC: Continuity of the value function.

It is stronger than normal controllability (Barles-Chasseigne)

Junction condition

- Supersolution type

$$-\partial_t u(t,o) + H^+(\nabla_x u(t,o)) \geq 0, \quad \forall t \in (0,T)$$

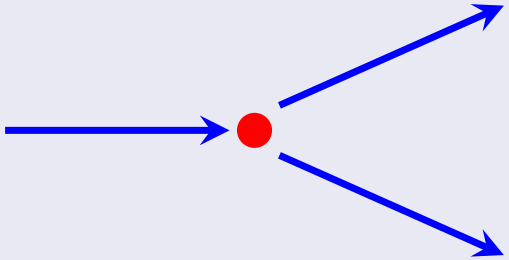
- Subsolution type

$$-\partial_t u(t,o) + H^+(\nabla_x u(t,o)) \leq 0, \quad \forall t \in (0,T)$$

Here

$$H^+(p) = \max_{i=1,\dots,p} \sup_{v \in F_i(o)} \{-\langle v e_i, p\rangle - L_i(o,v) \mid v \geq 0\}$$

Lower semicontinuous case



Goal of this talk

Find junction conditions to make ϑ the unique solution of (HJ)...

... in the **bilateral viscosity sense**.

Bilateral Viscosity solutions

Definition

An l.s.c. function $u : [0,T] \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ *is called a bilateral viscosity solution to* (HJ) *if*

- u *is a supersolution of*

$$-\partial_t u(t,x) + H_i(x, \nabla_x u(t,x)) = 0, \quad \forall (t,x) \in (0,T) \times \mathcal{M}_i,$$

- u *is a supersolution of*

$$\partial_t u(t,x) - H_i(x, \nabla_x u(t,x)) = 0, \quad \forall (t,x) \in (0,T) \times \mathcal{M}_i,$$

- *the following terminal conditions hold:*

$$\liminf_{t \to T^-,\ \tilde{x} \to x} u|_{\mathcal{M}_i}(t,\tilde{x}) = u(T,x) = g_i(x), \quad \forall x \in \mathcal{M}_i.$$

$$\liminf_{t \to 0^-,\ \tilde{x} \to x} u|_{\mathcal{M}_i}(t,\tilde{x}) = u(0,x), \quad \forall x \in \mathcal{M}_i.$$

Bilateral Viscosity solutions

Simple fact

$u : [0,T] \times \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is a bilateral viscosity solution iff

$$-\theta + H_i(x,\zeta) = 0, \quad \forall (t,x) \in (0,T) \times \mathcal{M}_i,\ \forall (\theta,\zeta) \in \partial_V u(x,t),$$

and the following terminal conditions hold:

$$\liminf_{t \to T^-,\ \tilde{x} \to x} u|_{\mathcal{M}_i}(t,\tilde{x}) = u(T,x) = g_i(x), \quad \forall x \in \mathcal{M}_i.$$

$$\liminf_{t \to 0^-,\ \tilde{x} \to x} u|_{\mathcal{M}_i}(t,\tilde{x}) = u(s,x), \quad \forall x \in \mathcal{M}_i.$$

where

$$\partial_V u(t,x) := \left\{ (\theta,\zeta) \in \mathbb{R} \times \mathbb{R}^N \,\middle|\, \begin{array}{c} \exists \varphi \in \mathcal{C}^1,\ \partial_t \varphi(t,x) = \theta,\ \nabla_x \varphi(t,x) = \zeta \\ u - \varphi \text{ has a local minimum at } (t,x) \end{array} \right\}$$

Information at the junctions

In our model:

$$g(x) = \begin{cases} g_i(x) & x \in \mathcal{M}_i \\ \min\{g_1(o), \dots, g_p(o)\} & x = o \end{cases}$$

$$L(x, v) = \begin{cases} L_i(x, v) & x \in \mathcal{M}_i, \\ \inf\{L_1(o, v), \dots, L_p(o, v)\} & x = o. \end{cases}$$

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Optimal control problem

$$\vartheta(t, x) := \left\{ \begin{array}{l} \text{Minimize } \displaystyle\int_t^T L(y(s), \dot{y}(s))ds + g(y(T)) \\ \text{over all } y \in AC([0, T]) \text{ s.t. } y(t) = x \text{ and} \\ \dot{y}(s) \in \begin{cases} F_i(y(s)) & \text{a.e. when } y(s) \in \mathcal{M}_i \\ \{0\} & \text{a.e. when } y(s) = o. \end{cases} \end{array} \right.$$

Existence of solutions and LSC

Proposition (Graber, CH, Zidani)

Under standard assumptions on the data. If

- $\exists i \in \{1, \dots, p\}, \quad F_i(o) \cap [0, +\infty) \neq \emptyset$
- $p \mapsto H_i(x, p)$ *is finite and convex for each* $x \in \overline{\mathcal{M}}_i$,
- $L_i(o, v) \geq L(o, 0), \quad \forall i \in \{1, \dots, p\}, \ \forall v \in F_i(o)$.

Then,

- *for every* $(t, x) \in [0, T) \times \mathcal{K}$ *there exists an optimal trajectory.*
- *the Value Function* ϑ *is Real-valued and l.s.c. on* $[0, T] \times \mathcal{K}$.

Proof.

Classical compactness arguments + structure of networks. □

Minimality as supersolution

Similarly to Cardaliaguet, Quincampoix, Saint-Pierre '97.

Theorem (Graber, CH, Zidani)

Under the same assumptions as in the preceding proposition, the Value Function ϑ is the smallest bilateral viscosity solution to (HJ) *satisfying*

- *a supersolution-type condition:*

$$\forall t \in (0,T), \qquad -\partial u(t,o) + H^+(\nabla_x u(t,o)) \geq 0.$$

- *and the terminal conditions*

$$\liminf_{t \to T^-,\ x \to o} u(t,x) = u(T,o) = g(o).$$

$$\liminf_{t \to 0^-,\ x \to 0} u(t,o) = u(0,o).$$

Recall that $H^+(p) = \max_{i=1,\dots,p} \sup_{v \in F_i(o)} \{-\langle v e_i, p\rangle - L_i(o,v) \mid v \geq 0\}$

Sketch of the proof

- Consider a "convexified problem" on the junction if

$$0 \in F_i(o)$$

- Construct a trajectory such that

$$u(T, y(T)) + \int_0^T \ell(y(s), \dot{y}(s))ds \leq u(t,x) \;\; \Rightarrow \vartheta(t,x) \leq u(t,x).$$

Weak invariance/Viability argument on the junction:

$$\exists \tau \mapsto \gamma(t) := \left(\tau, y(\tau), \int_t^\tau L(y(s), \dot{y}(s))ds\right)$$

so that

$$(\gamma(\tau), u(t,x)) \in \mathrm{epi}(U), \quad \forall \tau \in [t,T],$$

where $U(t,x,z) = u(t,x) + z$.

Characterization of the Value Function

So far we have if u is a bilateral viscosity solution + Junction conditions, then

$$\vartheta(t,x) \leq u(t,x), \quad \forall (t,x) \in [0,T] \times \mathcal{K}.$$

For "=", let us assume $x \mapsto L_i(x,v)$ is locally Lipschitz continuous.

Maximality as subsolution

Theorem (Graber, CH, Zidani)

Under the same assumptions as in the preceding theorem, the Value Function ϑ is the largest bilateral viscosity solution to (HJ) *satisfying the terminal conditions and*

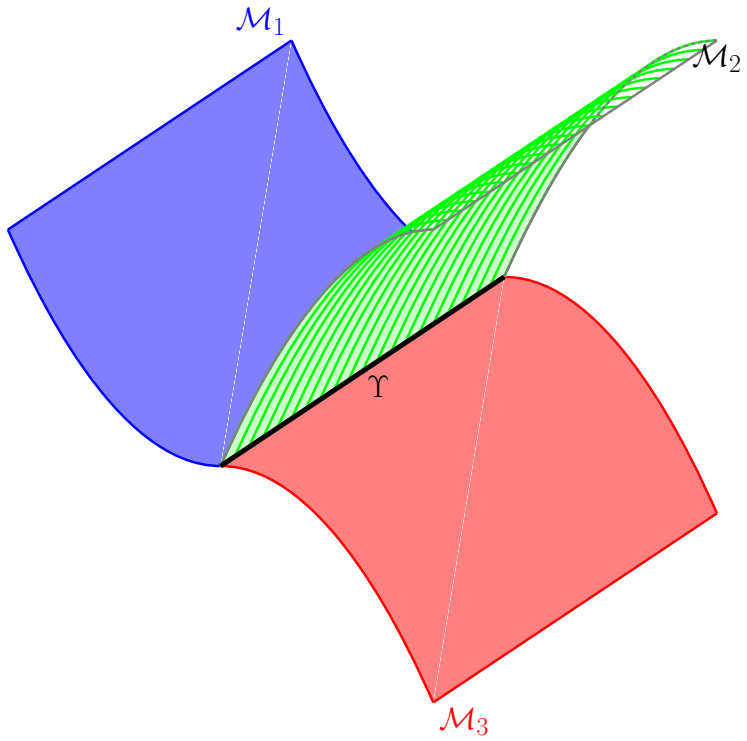
- *a subsolution-type condition:*

$$\forall t \in (0,T), \qquad -\partial u(t,o) + H^-(\nabla_x u(t,o)) \leq 0,$$

Here

$$H^-(p) = \sup_{i=1,\dots,p} \max_{v \in F_i(o)} \{-\langle v e_i, p\rangle - L_i(o,v) \mid v \leq 0\}$$

Extension to higher dimension networks



A set $\mathcal{K}$ is called a (d-dimensional) network if:

$$\mathcal{K} = \Upsilon \cup \mathcal{M}_1 \cup \ldots \cup \mathcal{M}_p,$$

where :

- $\Upsilon = \bigcap_{i=1}^{p} \overline{\mathcal{M}}_i$ is the junction, a $(d-1)$-dimensional manifold
- $\mathcal{M}_1, \ldots, \mathcal{M}_p$ are d-dimensional manifolds

Information at the junctions

In our model:

$$g(x) = \begin{cases} g_i(x) & x \in \mathcal{M}_i \\ \min\{g_1(x), \dots, g_p(x)\} & x \in \Upsilon \end{cases}$$

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Simplification

$$\Upsilon = \mathrm{span}\{e_1, \ldots, e_{d-1}\} \times \{0\}, \quad \text{with } |e_0| = \ldots = |e_{d-1}| = 1$$

and each

$$\mathcal{M}_i = \Upsilon \times (0, +\infty) e_{i,d}, \quad \text{with } |e_{i,d}| = 1.$$

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$$\mathcal{M}_i = \Upsilon \times (0, +\infty) e_{i,d}, \quad \text{with } |e_{i,d}| = 1.$$

Under these circumstances

$$\mathcal{T}_\Upsilon(x) = \Upsilon \quad \text{and} \quad \mathcal{T}^B_{\overline{\mathcal{M}}_i}(x) = \overline{\mathcal{M}}_i = \Upsilon \times [0, +\infty) e_{i,d}$$

and

$$F_i(x) = F_i^-(x) \oplus F_i^0(x) \oplus F_i^-(x)$$

with $F_i^+(x) \subseteq \mathcal{M}_i$, $F_i^0(x) \subseteq \Upsilon$ and $-F_i^-(x) \subseteq \mathcal{M}_i$.

Extensions of hypotheses

- $\exists i \in \{1, \ldots, p\}, \quad F_i(o) \cap [0, +\infty) \neq \emptyset$

$$\Downarrow$$

$$\exists i \in \{1, \ldots, p\}, \quad F_i^+(x) \cup F_i^0(x) \neq \emptyset, \quad \forall x \in \Upsilon$$

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- $L_i(o, v) \geq L(o, 0), \quad \forall i \in \{1, \ldots, p\}, \ \forall v \in F_i(o)$.

$$\Downarrow$$

In the case $F^0(x) := F_1^0(x) \cup \ldots \cup F_p^0(x) \neq \emptyset$,
setting $F(x) := F_1^0(x) \cup \ldots \cup F_p^0(x)$

$$\begin{cases} \forall x \in \Upsilon, \ \lambda \in [0, 1] \text{ and } v, \tilde{v} \in F(x), \ v_\lambda := \lambda v + (1 - \lambda)\tilde{v} \\ v_\lambda \in \Upsilon \ \Rightarrow \ v_\lambda \in F(x) \text{ and } \lambda L(x, v) + (1 - \lambda) L(x, \tilde{v}) \geq L(x, v_\lambda) \end{cases}$$

Minimality as supersolution

Theorem (Graber, CH, Zidani)

Under the same assumptions as in the preceding proposition, the Value Function ϑ is the smallest bilateral viscosity solution to (HJ) *satisfying*

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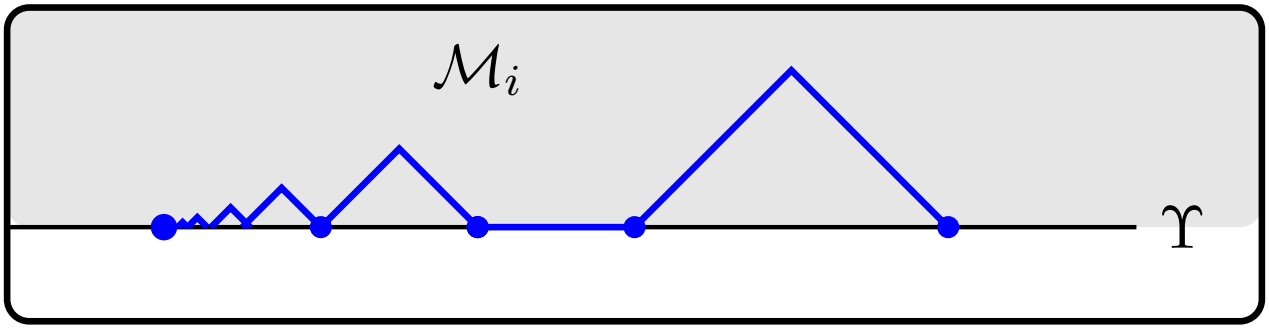
Where

$$H^+(x,p) = \max_{i=1,\dots,p} \sup_{v \in F_i^+(x) \cup F_i^0(x)} \{-\langle v, p\rangle - L_i(x,v)\}$$

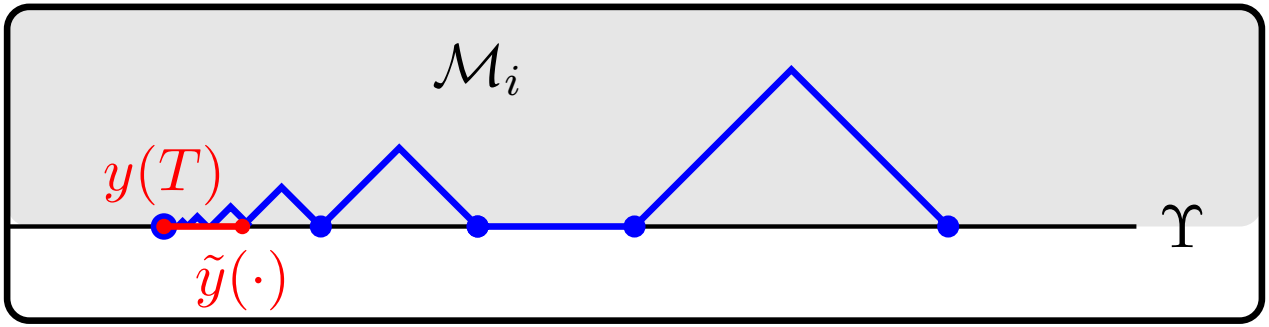
Suboptimality part: new hypotheses

- $\forall i \in \{1, \ldots, \}$, $x \mapsto F_i^0(x), F_i^-(x)$, are locally Lipschitz continuous.
- $\exists \delta > 0$ such that $\mathbb{B}(x, \delta) \cap \Upsilon \subseteq (F_1^0(x) \cup \ldots \cup F_p^0(x))$.

Chattering trajectory:



Approximated trajectory:



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$$\forall (t,x) \in (0,T) \times \Upsilon, \qquad -\partial u(t,x) + H^-(x, \nabla_x u(t,x)) \leq 0,$$

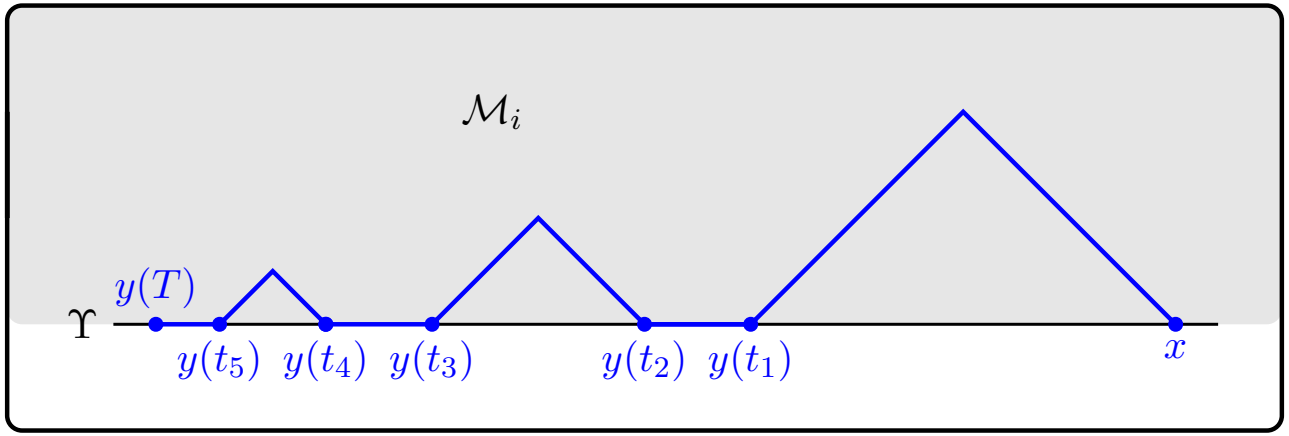
Where

$$H^-(x,p) = \max_{i=1,\dots,p} \sup_{v \in F_i(x)} \{-\langle v, p\rangle - L_i(x,v) \mid -v \in \overline{\mathcal{M}}_i\}$$

Suppose that $y(s) \in \mathcal{K}$, $\forall s \in [0, T]$ and there exists a partition of $[0, T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

so that $\forall l \in \{0, \ldots, n\}$, either $y(s) \in \mathcal{M}_i$ or $y(s) \in \Upsilon$ all along (t_l, t_{l+1}).



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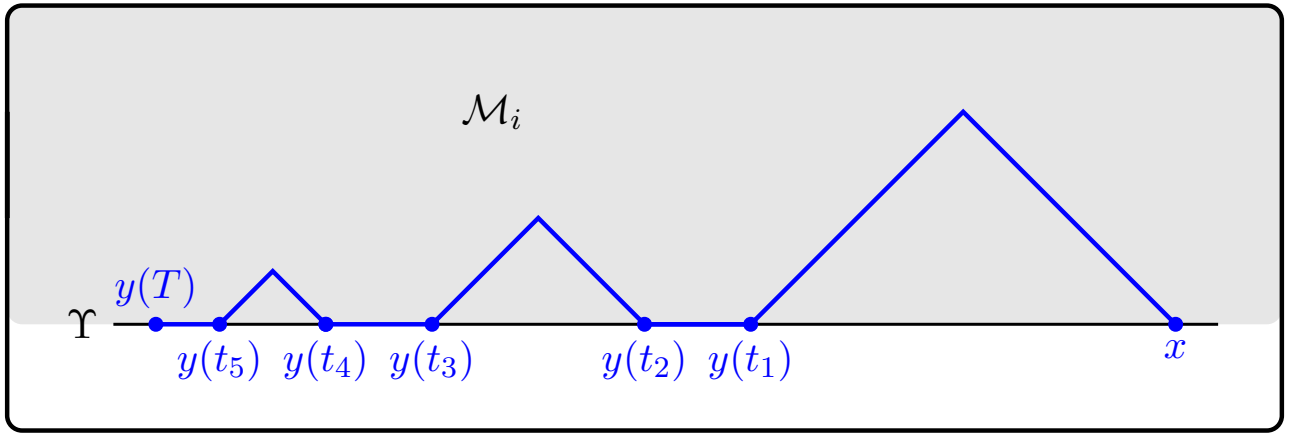

The HJB together with junction conditions imply

$$u(t_l, y(t_l)) \leq u(t_{l+1}, y(t_{l+1})) + \int_{t_l}^{t_{l+1}} \ell(y(s), \dot{y}(s)) ds, \quad \forall l \in \{0, \ldots, n\}.$$

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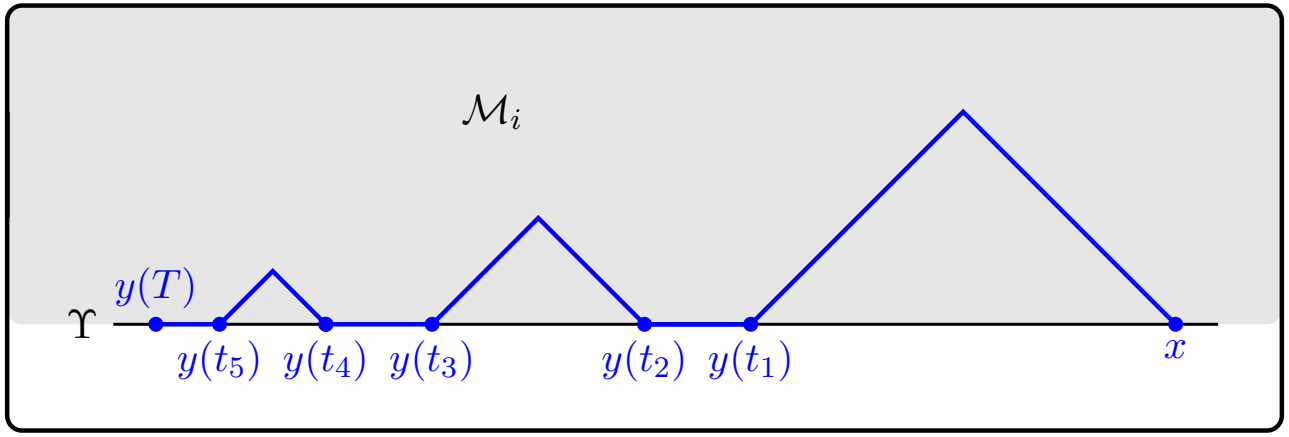

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In other words

$$u(t, x) \leq u(T, y(T)) + \int_0^T \ell(y(s), \dot{y}(s)) ds \;\Rightarrow u(t, x) \leq \vartheta(t, x).$$

Suppose that $y(s) \in \mathcal{K}$, $\forall s \in [0,T]$ and there exists a partition of $[0,T]$

$$\pi = \{0 = t_0 < t_1 < \ldots < t_n < t_{n+1} = T\}$$

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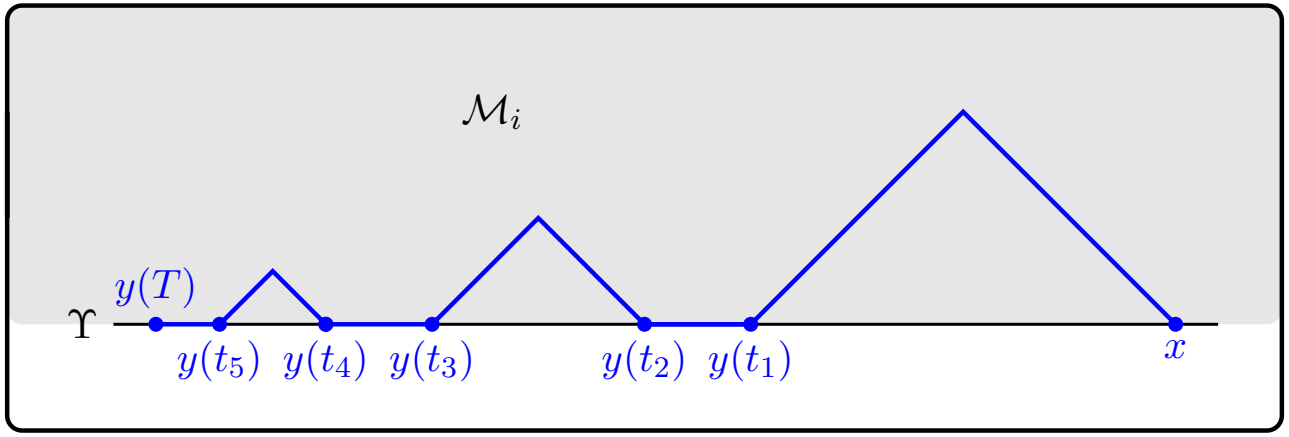

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$$u(t_l, y(t_l)) \leq u(t_{l+1}, y(t_{l+1})) + \int_{t_l}^{t_{l+1}} \ell(y(s), \dot{y}(s))ds, \quad \forall l \in \{0,\ldots,n\}.$$

In other words

$$u(t,x) \leq u(T, y(T)) + \int_0^T \ell(y(s), \dot{y}(s))ds \;\Rightarrow u(t,x) \leq \vartheta(t,x).$$

Many thanks for your attention!