



THE VANISHING DISCOUNT PROBLEM AND GENERALIZED MATHER MEASURES

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arXiv:1603.01051

at Université de Rennes 1, 05/30/2016

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VANISHING DISCOUNT PROBLEM

We consider the second-order PDE

$$\lambda v(x) + F(x, Dv(x), D^2v(x)) = 0 \quad \text{in } \mathbb{T}^n.$$

Here

$$\left\{ \begin{array}{l} v = v^\lambda \text{ denotes the unknown function on } \mathbb{T}^n \\ \lambda > 0 \text{ is a given constant,} \\ F \text{ is a given function of } (x, Dv(x), D^2v(x)). \end{array} \right.$$

The functions v and $F(\cdot, p, X)$ are identified periodic functions on \mathbb{R}^n with period \mathbb{Z}^n .

$$Dv = (v_{x_1}, \dots, v_{x_n}), \quad D^2v = (v_{x_i x_j})_{1 \leq i, j \leq n}.$$

Problem: asymptotic behavior of v^λ as $\lambda \rightarrow 0$.

CLASS OF PDES

Hypotheses:

(F1) F has the form

$$F(x, p, X) = \sup_{\alpha \in \mathcal{A}} (-\operatorname{tr} a(x, \alpha)X - b(x, \alpha) \cdot p - L(x, \alpha))$$

$$\text{for } (x, p, X) \in \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{S}^n,$$

where \mathcal{A} is a σ -compact, locally compact metric space ($\neq \emptyset$), \mathbb{S}^n denotes the space of $n \times n$ real symmetric matrices,

$$a \in C(\mathbb{T}^n \times \mathcal{A}, \mathbb{S}_+^n), \quad b \in C(\mathbb{T}^n \times \mathcal{A}, \mathbb{R}^n), \quad L \in C(\mathbb{T}^n \times \mathcal{A}, \mathbb{R}),$$

and \mathbb{S}_+^n denotes the subset of \mathbb{S}^n consisting of non-negative matrices.

(F2) F is a continuous function on $\mathbb{T}^n \times \mathbb{R}^n \times \mathbb{S}^n$.

Property: F is degenerate elliptic. That is,

$$X \leq Y \implies F(x, p, X) \geq F(x, p, Y)$$

Property: F is a convex function. More precisely,

$$(p, X) \mapsto F(x, p, X) \text{ is convex on } \mathbb{R}^n \times \mathbb{S}^n.$$

Notation : $\mathcal{L}u = \mathcal{L}u(x, \alpha) := -\operatorname{tr} a D^2 u - b \cdot Du.$

ERGODIC PROBLEM

$$(DP) \quad \lambda v^\lambda + F(x, Dv^\lambda, D^2v^\lambda) = 0 \quad \text{in } \mathbb{T}^n.$$

A classical observation regarding the behavior of the solutions v^λ , as $\lambda \rightarrow 0+$, is the following (P.-L. Lions-G. Papanicolaou-S. R. S. Varadhan).

Under suitable assumptions, for some constant $c \in \mathbb{R}$ and function $u \in C(\mathbb{T}^n)$, as $\lambda \rightarrow 0+$, we have

$$\begin{cases} -\lambda v^\lambda(x) \rightarrow c & \text{uniformly on } \mathbb{T}^n, \\ v^\lambda(x) - m^\lambda \rightarrow u(x) & \text{uniformly on } \mathbb{T}^n \text{ along a subsequence,} \end{cases}$$

where m_λ is chosen as $m^\lambda = \min_{\mathbb{T}^n} v^\lambda$, for instance. Furthermore, the pair (u, c) is a solution of

$$(EP) \quad F(x, Du(x), D^2u(x)) = c \quad \text{in } \mathbb{T}^n.$$

The problem

$$(EP) \quad F(x, Du(x), D^2u(x)) = c \quad \text{in } \mathbb{T}^n.$$

is called the **ergodic problem** or **additive eigenvalue problem**. Here the problem is to find a pair (u, c) of a solution u of PDE (EP) and a constant c .

Main question: In recent years there has been a growing interest in the question if the whole family $\{v^\lambda - m^\lambda\}_{\lambda>0}$ converges to a function as $\lambda \rightarrow 0+$.

A few previous work:

1) A. Davini, A. Fathi, R. Iturriaga, M. Zavidovique,
First-order HJ equation on \mathbb{T}^n (closed manifold), with coercive and convex Hamiltonians.

2) E. S. Al-Aidarous, E. O. Alzahrani, HI, A. M. M. Younas,
First-order HJ equation with the Neumann type BC, with coercive and convex Hamiltonian.

3) H. Mitake, H. V. Tran
Viscous HJ equation on \mathbb{T}^n , with smooth, coercive and convex Hamiltonian.

• Use of **Mather measures** , the **adjoint method** due to L. C. Evans.

CLASSICAL OBSERVATIONS

Two more hypotheses:

(CP) $\left\{ \begin{array}{l} \text{The comparison principle holds for (DP). More precisely,} \\ \text{if } \lambda > 0 \text{ and if } v \in \text{USC}(\mathbb{T}^n), w \in \text{LSC}(\mathbb{T}^n) \text{ are a sub-} \\ \text{solution and a supersolution of (DP), respectively, then} \\ v \leq w \text{ in } \mathbb{T}^n. \end{array} \right.$

Proposition 1

Assume (F1), (F2) and (CP). Let $\lambda > 0$. Problem (DP) has a unique solution in $C(\mathbb{T}^n)$.

(EC) $\left\{ \begin{array}{l} \text{For every } \lambda > 0, \text{ there exists a solution } v^\lambda \in C(\mathbb{T}^n) \text{ of} \\ \text{(DP), and the family } \{v^\lambda\}_{\lambda>0} \text{ is equi-continuous on } \mathbb{T}^n. \end{array} \right.$

Proposition 2


Assume (F1), (F2), (CP), and (EC). (i) Problem (EP) has a solution $(u, c) \in C(\mathbb{T}^n) \times \mathbb{R}$, and the constant c is determined uniquely. (ii) For $\lambda > 0$, let $v^\lambda \in C(\mathbb{T}^n)$ be a unique solution of (DP), Then

$$c = - \lim_{\lambda \rightarrow 0^+} \lambda v^\lambda(x) \text{ in } C(\mathbb{T}^n),$$

and, for any sequence $\{\lambda_j\}_{j \in \mathbb{N}} \subset (0, \infty)$ converging to zero, there exists its subsequence, which is denoted by the same symbol, such that $\{v^{\lambda_j} - \min_{\mathbb{T}^n} v^{\lambda_j}\}_{j \in \mathbb{N}}$ converges in $C(\mathbb{T}^n)$. Moreover, the pair of the function

$$u(x) := \lim_{j \rightarrow \infty} (v^{\lambda_j}(x) - \min_{\mathbb{T}^n} v^{\lambda_j}) \in C(\mathbb{T}^n),$$

and the constant c is a solution of (EP).

If $(u, c) \in C(\mathbb{T}^n) \times \mathbb{R}$ is a solution of (EP), then the constant c is called a **critical value** or an additive eigenvalue. 

MAIN THEOREM

We introduce two more hypotheses.

(AC) the set \mathcal{A} is compact.

For simplicity, we write $F[u](x)$ for $F(x, Du(x), D^2u(x))$.

For any $\phi \in C(\mathbb{T}^n \times \mathcal{A})$, we consider the function defined by

$$F_\phi(x, p, X) = \sup (-\operatorname{tr} a(x, \alpha)X - b(x, \alpha) \cdot p - \phi(x, \alpha)),$$

and the problem

$$(DP_\phi) \quad \lambda v(x) + F_\phi[v] = 0 \quad \text{in } \mathbb{T}^n.$$

(CP') $\left\{ \begin{array}{l} \text{The comparison principle holds for } (DP_\phi). \text{ More precisely,} \\ \text{let } \lambda > 0, \phi \in C(\mathbb{T}^n \times \mathcal{A}), \text{ and } U \text{ be any open subset of} \\ \mathbb{T}^n. \text{ If } v, w \in C(U) \text{ are a subsolution and a supersolution} \\ \text{of } \lambda u + F_\phi[u] = 0 \text{ in } U, \text{ respectively, and } v \leq w \text{ on } \partial U, \\ \text{then } v \leq w \text{ in } U. \end{array} \right.$

Main theorem

Assume (F1), (F2), (AC), (CP'), and (EC). Let c be the critical value of (EP) and, for each $\lambda > 0$, let $v^\lambda \in C(\mathbb{T}^n)$ be the unique solution of (DP). Then, the family $\{v^\lambda + \lambda^{-1}c\}_{\lambda>0}$ converges to a function u in $C(\mathbb{T}^n)$ as $\lambda \rightarrow 0$. Furthermore, the pair (u, c) is a solution of (EP).

The comparison principle (CP') is strong enough to guarantees the conclusion of Proposition 2.

VISCOSITY MATHER MEASURES

We introduce sort of a generalized Mather measure, that applies to second-order degenerate elliptic PDEs. This generalization is inspired by

D. Gomes, Duality principle for fully nonlinear ... 2005.

Henceforth we assume that the **critical value c is equal to zero** .

This can be always realized by replacing F by $F - c$ as well as v^λ by $v^\lambda + \lambda^{-1}c$.

Let $\phi \in C(\mathbb{T}^n \times \mathcal{A})$ and consider subsolutions $u \in C(\mathbb{T}^n)$ of the PDE

$$(EP_\phi) \quad F_\phi[u] = 0 \quad \text{in } \mathbb{T}^n.$$

We denote the set of all such pairs (ϕ, u) by $\mathcal{F}_\pi(0)$. The set $\mathcal{F}_\pi(0)$ is **positively homogeneous** , that is,

$$t > 0, (\phi, u) \in \mathcal{F}_\pi(0) \implies t(\phi, u) \in \mathcal{F}_\pi(0).$$

This set is also **convex**, thanks to the convexity of $(p, X) \mapsto F(x, p, X)$.

Lemma 1

Under hypotheses (F1), (F2) and (CP'), the set $\mathcal{F}_\pi(0)$ is a convex set in $C(\mathbb{T}^n \times \mathcal{A})$.

Thus, $\mathcal{F}_\pi(0)$ is a **convex cone** with vertex at the origin.

Consider the **dual cone** $\mathcal{F}_\pi'(0)$ of $\mathcal{F}_\pi(0)$ in the space of all Radon measures. That is, a Radon measure μ is in $\mathcal{F}_\pi'(0)$ if and only if

$$0 \leq \langle \mu, \phi \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(0),$$

where

$$\langle \mu, \phi \rangle := \int_{\mathbb{T}^n \times \mathcal{A}} \phi(x, \alpha) \mu(dx d\alpha) \quad (C^*(\mathbb{T}^n \times \mathcal{A}) \overset{\text{duality}}{\longleftrightarrow} C(\mathbb{T}^n \times \mathcal{A})).$$

We set

$$\mathcal{P}_\pi(0) = \{ \mu \in \mathcal{F}_\pi'(0) : \mu \text{ is a } \mathbf{probability} \text{ measure on } \mathbb{T}^n \times \mathcal{A} \}.$$

The next claim ensures the existence of “Mather measures”.

Theorem 1

Assume (F1), (F2), (AC), (CP') and (EC). Also, assume that $c = 0$. Then,

$$\min_{\mu \in \mathcal{P}_\pi(0)} \langle \mu, L \rangle = 0.$$

Here the role of (EC) is to guarantee that (EP) has a solution.

We call $\mu \in \mathcal{P}_\pi(0)$ a **viscosity Mather measure** if it attains the minimum value of the left hand side of the identity above.

The key idea for the proof of the identity above is to use **Sion's minimax theorem**, which differs from the use of **the convex duality** by Diogo Gomes.

A crucial property of $\mathcal{P}_\pi(0)$ is introduced here as the dual cone of $\mathcal{F}_\pi(0)$, which corresponds to the closedness property of Mather measures. In the work of D. Gomes, he chooses, for $\mathcal{F}_\pi(0)$, the linear space of the pairs $(\phi, \psi) \in C(\mathbb{T}^n \times \mathcal{A}) \times C^2(\mathbb{T}^n)$, where

$$\phi : (x, \alpha) \mapsto \mathcal{L}\psi = -\operatorname{tr} a(x, \alpha) D^2\psi(x) - b(x, \alpha) \cdot D\psi(x).$$

Note that this pair (ϕ, ψ) belongs to $\mathcal{F}_\pi(0)$. Indeed,

$$F_\phi[\psi] = \max_{\alpha \in \mathcal{A}} \{ \mathcal{L}\psi(x, \alpha) - \phi(x, \alpha) \} = 0.$$

In the approach by D. Gomes, the dual cone property can be stated as

$$\langle \mu, \mathcal{L}\psi \rangle = 0 \quad \forall \psi \in C^2(\mathbb{T}^n).$$

This explains why we call our measures “viscosity” Mather measures.

We have a theorem, similar to the above, for discount problem (DP). We fix $(z, \lambda) \in \mathbb{T}^n \times (0, \infty)$. Define

$\mathcal{F}_\pi(\lambda) \subset C(\mathbb{T}^n \times \mathcal{A}) \times C(\mathbb{T}^n)$ by

$\mathcal{F}_\pi(\lambda) = \{(\phi, u) \in C(\mathbb{T}^n \times \mathcal{A}) \times C(\mathbb{T}^n) : u \text{ is a subsolution of } (DP_\phi)\}$,

and $\mathcal{P}_\pi(z, \lambda)$ as the set of Radon probability measures μ on $\mathbb{T}^n \times \mathcal{A}$ having the property

$$0 \leq \langle \mu, \phi - \lambda u(z) \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(\lambda).$$

Theorem 2

Assume (F1), (F2), (AC) and (CP'). Let $\lambda \geq 0$ and $v^\lambda \in C(\mathbb{T}^n)$ be the (unique) solution of (DP). Then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_\pi(z, \lambda)} \langle \mu, L \rangle.$$

Representation formula for solutions of (DP).

We call a minimizer $\mu \in \mathcal{P}_\pi(z, \lambda)$ of the following minimization problem a **discounted Mather measure**

$$\min_{\mu \in \mathcal{P}_\pi(z, \lambda)} \langle \mu, L \rangle.$$

Following the argument by Davini-Fathi-Iturriaga-Zavidovique and using Theorems 1 and 2, the proof of Main theorem is now easy.

FURTHER REMARKS

1. The case when \mathcal{A} is non-compact. We assume that

$$(L) \quad L = +\infty \quad \text{at infinity.}$$

We introduce

$$\Phi^+ = \{tL + \chi : t > 0, \chi \in C(\mathbb{T}^n)\}.$$

We replace $\mathcal{F}_\pi(0)$ (resp., $\mathcal{F}_\pi(z, \lambda)$) by the set of

$(\phi, u) \in \Phi^+ \times C(\mathbb{T}^n)$ such that u is a subsolution of (EP_ϕ) (resp., (DP_ϕ)). Let \mathcal{P}_L denote the space of Radon probability measures μ such that L is integrable on $\mathbb{T}^n \times \mathcal{A}$ with respect to μ . We replace $\mathcal{P}_\pi(0)$ (resp., $\mathcal{P}_\pi(z, \lambda)$) by the set of $\mu \in \mathcal{P}_L$ with the property

$$0 \leq \langle \mu, \phi \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(0)$$

$$\left(\text{resp., } 0 \leq \langle \mu, \phi - \lambda u(z) \rangle \quad \text{for all } (\phi, u) \in \mathcal{F}_\pi(z, \lambda) \right).$$

(CP'') $\left\{ \begin{array}{l} \text{The comparison principle holds for } \lambda u + F[u] = \eta \text{ in } \\ \mathbb{T}^n, \text{ where } \eta \in C(\mathbb{T}^n). \text{ More precisely, let } \lambda > 0 \text{ and } U \\ \text{be any open subset of } \mathbb{T}^n. \text{ If } v, w \in C(U) \text{ are a subso-} \\ \text{lution and a supersolution of } \lambda u + F(x, Du, D^2u) = \eta \\ \text{in } U, \text{ respectively, and } v \leq w \text{ on } \partial U, \text{ then } v \leq w \text{ in} \\ U. \end{array} \right.$

Theorem 3

Assume (F1), (F2), (L), (CP'') and (EC). Assume that the critical value c is zero. Then,

$$\min_{\mu \in \mathcal{P}_\pi(0)} \langle \mu, L \rangle = 0.$$

Fix $(z, \lambda) \in \mathbb{T}^n \times (0, \infty)$ and let v^λ be the solution of (DP). Then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_\pi(z, \lambda)} \langle \mu, L \rangle.$$

2. Neumann problem. We replace \mathbb{T}^n by a bounded (smooth) domain $\Omega \subset \mathbb{R}^n$ and consider the **Neumann problem**

$$(DN) \quad \begin{cases} \lambda u + F[u] = 0 & \text{in } \Omega, \\ \gamma \cdot Du = g & \text{on } \partial\Omega. \end{cases}$$

Assume that $g \in C(\partial\Omega)$ and $\gamma : \partial\Omega \rightarrow \mathbb{R}^n$ is oblique to the boundary, that is,

$$\gamma \cdot \nu > 0 \quad \text{on } \partial\Omega,$$

where ν is the outward unit normal vector to Ω .

For $(\phi, \psi) \in \Phi^+ \times C(\partial\Omega)$, we consider the problem

$$(DN_{\phi, \psi}) \quad \begin{cases} \lambda u + F_{\phi}[u] = 0 & \text{in } \Omega, \\ \gamma \cdot Du = \psi & \text{on } \partial\Omega. \end{cases}$$

We define

$$\mathcal{F}_N(\lambda) = \{(\phi, \psi, u) : u \text{ is a subsolution of } (DN_{\phi, \psi})\}.$$

Here, λ can be equal to zero.

Define $\mathcal{P}_N(z, \lambda)$ (also, written as $\mathcal{P}_N(0)$ when $\lambda = 0$) by the condition that $(\mu_1, \mu_2) \in \mathcal{P}_N(z, \lambda)$ if and only if

$$\begin{cases} \mu_1 \in \mathcal{R}_L, & \mu_2 \in \mathcal{R}_{\partial\Omega} \text{ (the Radon measures on } \partial\Omega), \\ \mu_i \geq 0 \quad i = 1, 2, & \mu_1(\bar{\Omega} \times \mathcal{A}) = 1, \\ 0 \leq \langle \mu_1, \phi - \lambda u(z) \rangle + \langle \mu_2, \psi \rangle & \forall (\phi, \psi, u) \in \mathcal{F}_N(\lambda). \end{cases}$$

Theorem 4

Assume (F1), (F2), (L), (CPN'') and (EC). Assume that the critical value c_N is zero. Then,

$$\min_{\mu \in \mathcal{P}_N(0)} \int_{\mathbb{T}^n \times \mathcal{A}} L(x, \alpha) \mu(dx d\alpha) = 0.$$

Fix $(z, \lambda) \in \mathbb{T}^n \times (0, \infty)$ and let v^λ be the solution of (DN). Then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_N(z, \lambda)} (\langle \mu_1, L \rangle + \langle \mu_2, g \rangle).$$

3. Dirichlet problem. We consider the **Dirichlet problem**

$$(DD) \quad \begin{cases} \lambda u + F[u] = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Assume that $g \in C(\partial\Omega)$. The interpretation of (DD) is formally as follows.

$$\begin{cases} \text{subsolution:} & u \leq g \text{ pointwise on } \partial\Omega, \\ \text{supersolution:} & u \geq g \text{ or } \lambda u + F[u] \geq 0 \text{ on } \partial\Omega. \end{cases}$$

For $(\phi, \psi) \in \Phi^+ \times C(\partial\Omega)$, we consider the problem

$$(DD_{\phi, \psi}) \quad \begin{cases} \lambda u + F_{\phi}[u] = 0 & \text{in } \Omega, \\ u = \psi & \text{on } \partial\Omega. \end{cases}$$

We define

$$\mathcal{F}_D(\lambda) = \{(\phi, \psi, u) : u \text{ is a subsolution of } (DD_{\phi, \psi})\}.$$

Here, $\lambda \geq 0$.

Define $\mathcal{P}_D(z, \lambda)$ (also, written as $\mathcal{P}_D(0)$ when $\lambda = 0$) by the condition that $(\mu_1, \mu_2) \in \mathcal{P}_D(z, \lambda)$ if and only if

$$\begin{cases} \mu_1 \in \mathcal{R}_L, & \mu_2 \in \mathcal{R}_{\partial\Omega} \text{ (the Radon measures on } \partial\Omega\text{),} \\ \mu_i \geq 0 \quad i = 1, 2, & \mu_1(\bar{\Omega} \times \mathcal{A}) + \mu_2(\partial\Omega) = 1, \\ 0 \leq \langle \mu_1, \phi - \lambda u(z) \rangle + \lambda \langle \mu_2, \psi - u(z) \rangle & \forall (\phi, \psi, u) \in \mathcal{F}_D(\lambda). \end{cases}$$

Theorem 5

Assume (F1), (F2), (L), (CPD'') and (EC). Then,

$$-c_D = \min_{\mu \in \mathcal{P}_D(0)} \langle \mu_1, L \rangle.$$

Fix $(z, \lambda) \in \mathbb{T}^n \times (0, \infty)$ and let v^λ be the solution of (DD). Then

$$\lambda v^\lambda(z) = \min_{\mu \in \mathcal{P}_D(z, \lambda)} (\langle \mu_1, L \rangle + \lambda \langle \mu_2, g \rangle).$$

(CPD'') = (CP'') with \mathbb{T}^n replaced by $\bar{\Omega}$.