Homogenization of a mean field game system in the small noise limit

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Joint work with
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Rennes, 31st May, 2016

Outline

1. Brief introduction on MFG systems
2. Homogenization problems for MFG
3. Analysis of the effective operators and effective limit system
4. A (partial) convergence result.

A brief introduction to Mean Field Games

The Mean Field Games model (MFG) was proposed by Lasry-Lions, and independently by Huang-Malhamé-Caines, in 2006.

Distinctive features of the model:

- The MFG theory is a model to describe interactions among a very large number of agents. Aim of MFG theory is to relate individual actions to mass behavior (fashion trends, Mexican wave, financial crisis, crowd dynamics,...).
- The MFG model has some analogies with Statistical Mechanics, where an external field (usually a statistics of some given physical quantity) influences the behavior of the particles. But in MFG theory the agent is not a black-box, since it can decide a strategy based on a set of preferences.
- The single agent by itself cannot influence the collective behavior, it can only optimize its own strategy. The mean field is given by the collective behavior of the population.

Model example

Consider a game with N rational and indistinguishable players. The i-th player's dynamics is

$$dX_t^i = -\alpha_t^i dt + \sqrt{2} dW_t^i, \qquad X_0^i = x^i \in \mathbb{T}^n$$

where W^i are independent Brownian motions, while α^i is the control chosen so to minimize the cost functional

$$\mathbb{E}\left\{ \int_0^t \left[\frac{|\alpha_s^i|^2}{2} + V\left(X_s^i, \frac{1}{N-1} \sum_{j \neq i} \delta_{X_s^j} \right) \right] ds + u_0\left(X_T^i \right) \right\}$$

The Nash equilibria are characterized by a system of $2N$ equations. BUT, as $N \to +\infty$, this system reduces to the following one:

MFG system

$$\begin{cases} -u_t - \Delta u + \frac{1}{2}|\nabla u|^2 = V(x,m) & (t,x) \in (0,T) \times \mathbb{T}^n \\ m_t - \Delta m + \operatorname{div}(m \nabla u) = 0 & (t,x) \in (0,T) \times \mathbb{T}^n \\ u(T,x) = u_0(x), \qquad m(0,x) = m_0(x) & x \in \mathbb{T}^n \end{cases}$$

where m_0 is the initial distribution of players: $m_0 \geq 0 \int_{\mathbb{T}^d} m_0 = 1$.

MFG system

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\begin{cases}
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Features of MFG system

- The first equation is a *backward* Hamilton-Jacobi-Bellman equation describing the expected value for an average player.
- the second equation is a *forward* Fokker-Planck equation describing the density m of the players.
- The operator in the FP equation is the adjoint of the linearized of the operator in the HJB equation.

Two possible regimes for the coupling

1) **nonlocal coupling**: $V : \mathbb{R}^n \times \mathcal{P}_1 \to \mathbb{R}$ smoothing in the space $\mathcal{P}_1$ of probability measures, e.g.: $V(x, m) = V(x, m \star k)$.
Uniqueness when $V(y, \cdot)$ is monotone increasing:

$$\int (V(x, m) - V(x, n))(m - n)dx > 0.$$

Existence of smooth solutions via Schauder theorem (Lasry, Lions).

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2) **local coupling**: $V(x, m(x,t))$.

- In our case $H = |p|^2$, solutions are smooth (Cardaliaguet, Lasry, Lions, Porretta)
- solutions are still smooth under some regularity-growth assumptions (Lasry, Lions, Gomes, Pimentel, Sanchez Morgado)
- Weak solutions of MFG system: second order (Porretta), first order (Cardaliaguet), second order degenerate (Cardaliaguet, Graber, Porretta, Tonon)

Asymptotic problems on MFG : convergence to steady states

$$(MFG)\begin{cases} -u_t - \Delta u + \frac{1}{2}|\nabla u|^2 = V(x,m)\,, & u(x,T) = u_0(x) \\ m_t - \Delta m - \mathrm{div}(m\nabla u) = 0 & m(x,0) = m_0(x). \end{cases}$$

Stationary ergodic system (steady states)

$$\begin{cases} \lambda - \Delta \overline{u} + \frac{1}{2}|\nabla \overline{u}|^2 = V(x,\overline{m})\,, & \int_{\mathbb{T}^n} \overline{u}(y)dy = 0 \\ -\Delta \overline{m} - \mathrm{div}(\overline{m}\nabla \overline{u}) = 0 & \int_{\mathbb{T}^n} \overline{m}(y)dy = 1 \end{cases}$$

Theorem

$$\frac{u(\cdot,tT)}{T} \to \lambda(1-t) \qquad u(\cdot,tT) - \int u(x,tT)dx \to \bar{u} \qquad in\ L^2(\mathbb{T}^n \times (0,1))$$

$$m(x,tT) \to \bar{m}(x) \qquad in\ L^p(\mathbb{T}^n \times (0,1))\ \ p < n+2/n.$$

(Cardaliaguet, Lasry, Lions, Porretta, both local and nonlocal coupling)

Basic References for MFG theory:

- Lasry-Lions, C.R. Math. Acad. Sci. Paris 343 (2006), 619-625.
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Homogenization problems for MFG systems

We are interested in the following asymptotic problem: the limit as $\varepsilon \to 0$

$$\begin{cases} -u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 = V\left(\frac{x}{\varepsilon}, m^\varepsilon\right) & (t,x) \in (0,T) \times \mathbb{T}^n \\ m_t^\varepsilon - \varepsilon\Delta m^\varepsilon - \operatorname{div}(m^\varepsilon \nabla u^\varepsilon) = 0 & (t,x) \in (0,T) \times \mathbb{T}^n \\ u^\varepsilon(x,T) = u_0(x) \qquad m^\varepsilon(x,0) = m_0(x) & x \in \mathbb{T}^n. \end{cases}$$

where we consider periodic boundary conditions: both $V(\cdot, m)$ and $V(\frac{\cdot}{\varepsilon}, m)$ are both $\mathbb{Z}^n$-periodic (we consider $\varepsilon^{-1} \in \mathbb{N}$).

Remark. More generally we can assume just that $V(\cdot, m)$ is $\mathbb{Z}^n$-periodic and consider the system in a bounded domain with Neumann boudary conditions.

Main assumptions

- $V(y,m) \in C^1(\mathbb{T}^n \times \mathbb{R})$ is bounded,
- $V(y,\cdot)$ is monotone increasing in m,
- $V(\cdot,m)$ is $\mathbb{Z}^n$-periodic in y, $\varepsilon^{-1} \in \mathbb{N}$,
- u_0 is smooth and periodic,
- $m_0 \geq 0$, m_0 is smooth, periodic and $\int_{\mathbb{T}^n} m_0(y)dy = 1$.

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Proposition

$\forall \varepsilon > 0$, there exists a unique smooth solution to

$$\begin{cases} -u^\varepsilon_t - \varepsilon\Delta u^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 = V\left(\frac{x}{\varepsilon}, m^\varepsilon\right) & (t,x) \in (0,T) \times \mathbb{T}^n \\ m^\varepsilon_t - \varepsilon\Delta m^\varepsilon - \mathrm{div}(m^\varepsilon \nabla u^\varepsilon) = 0 & (t,x) \in (0,T) \times \mathbb{T}^n \\ u^\varepsilon(x,T) = u_0(x) \qquad m^\varepsilon(x,0) = m_0(x) & x \in \mathbb{T}^n. \end{cases}$$

Two main problems

1) identification of the limit system:

- formal 2 scale (additive-multiplicative) asymptotic expansion

$$\begin{cases} u^\varepsilon(x,t) = u^0(x,t) + \varepsilon u(x/\varepsilon) \\ m^\varepsilon(x,t) = m^0(t,x)\left(m(x/\varepsilon) + \varepsilon m^2(x/\varepsilon)\right) \end{cases}$$

- solution of an ergodic MFG system (so called cell problem) to identify the limit operators.

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- solution of an ergodic MFG system (so called cell problem) to identify the limit operators.

2) Proof of convergence (in a suitable sense)

To make rigorous the formal asymptotic expansion

- two scale convergence (Nguetseng 89, Allaire 92)
- perturbed test function method (Evans 89, 92)

Only preliminary results, under strong assumptions (i.e. assuming that the asymptotic expansion holds).

Two scale expansion

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and we get the following (let $y = \frac{x}{\varepsilon}$)

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- terms in ε^{-1} in the FP:

$$-\Delta_y m - \mathrm{div}_y((\nabla_x u^0 + \nabla_y u)m) = 0$$

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- terms in ε^0 in the HJ:

$$-u_t^0 - \Delta_y u + \frac{1}{2}|\nabla_x u^0 + \nabla_y u|^2 - V(y, m^0 m) = 0$$

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We freeze x, t: given $P = \nabla u^0(x,t) \in \mathbb{R}^n$ and $\alpha = m^0(x,t) \geq 0$, find the unique constant $\bar{H}(P,\alpha)$ for which there exists a periodic solution to

$$\begin{cases} -\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y,\alpha m) = \bar{H}(P,\alpha) & y \in \mathbb{T}^n \\ -\Delta m - \mathrm{div}\,(m(\nabla u + P)) = 0 & y \in \mathbb{T}^n. \end{cases}$$

Proposition (cell problem)

Given $P \in \mathbb{R}^n$ and $\alpha \geq 0$, there exists a unique constant $\bar{H}$ such that

$$\begin{cases} -\Delta u + \frac{1}{2}|\nabla u + P|^2 - V(y, \alpha m) = \overline{H}(P, \alpha) & y \in \mathbb{T}^n \\ -\Delta m - \operatorname{div}\left(m\left(\nabla u + P\right)\right) = 0 & y \in \mathbb{T}^n \\ \int_{\mathbb{T}^n} u = 0 \qquad \int_{\mathbb{T}^n} m = 1 \end{cases}$$

admits a solution (u, m).

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- this solution is unique; $u \in C^{2,\gamma}$, $m \in W^{1,p}$ with $m \geq c > 0$;

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- this solution is unique; $u \in C^{2,\gamma}$, $m \in W^{1,p}$ with $m \geq c > 0$;
- $\bar{H}$ is decreasing in α and coercive in P:

$$\frac{|P|^2}{2} - \|V\|_\infty \leq \bar{H}(P, \alpha) \leq \frac{|P|^2}{2} + \|V\|_\infty;$$

- for any $\gamma \in (0, 1)$, $p \in (1, +\infty)$, the following maps are continuous

$$(P, \alpha) \to \bar{H}(P, \alpha) \in \mathbb{R}, \qquad (P, \alpha) \to (u, m) \in C^{1,\gamma} \times W^{1,p}.$$

Proof: based on [Cardaliaguet-Lasry-Lions-Porretta], [Bardi-Felequi].

Going back to the two scale expansion

- terms in ε^0 in the FP:

$$m^0(-\Delta_y m^2 - \mathrm{div}_y(m^2(\nabla_x u^0 + \nabla_y u))) =$$
$$-mm^0_t + m\mathrm{div}_x(m^0(\nabla_x u^0 + \nabla_y u)) + 2\nabla_x m_0 \cdot \nabla_y m.$$

So the solvability condition for the equation in m^2 gives

$$m^0_t - \mathrm{div}_x\left(m^0\left(\int_{\mathbb{T}^n} m(P+\nabla u)dy\right)\right) = 0$$

We denote with

$$\bar{b}(P,\alpha) = \int_{\mathbb{T}^n} m(P+\nabla u)dy.$$

Expected limit system

At the limit, we expect the following system

$$\begin{cases} -u^0_t + \bar{H}(\nabla u^0, m^0) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ m^0_t - \mathrm{div}(m^0 \bar{b}(\nabla u^0, m^0)) = 0, & (t,x) \in (0,T) \times \mathbb{R}^n \\ u^0(x,T) = u_0(x) \qquad m^0(x,0) = m_0(x) & x \in \mathbb{R}^n \end{cases}$$

where

- $\bar{H}(P,\alpha)$ and $\bar{b}(P,\alpha)$ are locally Lipschitz continuous
- $\bar{H}(P,\alpha)$ is coercive in P with quadratic growth (unif. in α)
- $\bar{H}(P,\alpha)$ is monotone decreasing in α

Questions. Existence of (weak) solutions? Uniqueness?
Even monotonicity of the system is not clear.

(Brief) discussion on other cases

Case of finite-not vanishing- noise

$$\begin{cases} -u^\varepsilon_t - \Delta u^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 = V\left(\frac{x}{\varepsilon}, m^\varepsilon\right) & (t,x) \in (0,T) \times \mathbb{T}^n, \\ m^\varepsilon_t - \Delta m^\varepsilon - \mathrm{div}(m^\varepsilon \nabla u^\varepsilon) = 0 & (t,x) \in (0,T) \times \mathbb{T}^n. \end{cases}$$

Formal 2 scale asymptotic expansion

$u^\varepsilon(x,t) = u^0(x,t) + \varepsilon^2 u(x/\varepsilon)$

$m^\varepsilon(x,t) = m^0(t,x)\left(m(x/\varepsilon) + \varepsilon^2 m^2(x/\varepsilon)\right)$

- cell problem decouples;
- $\Delta m = 0$, m periodic and mean 1; hence, $m \equiv 1$;
- expected limit system has a MFG structure with a monotone coupling:

$$\begin{cases} -u^0_t - \Delta u^0 + \frac{1}{2}|\nabla u^0|^2 = \int_{\mathbb{T}^n} V(y, m^0(x,t))dy \\ m^0_t - \Delta m^0 - \mathrm{div}(m^0 \nabla u^0) = 0. \end{cases}$$

A case of nonlocal coupling

$$\begin{cases} -u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + \frac{1}{2}|Du^\varepsilon|^2 = V\left(\frac{x}{\varepsilon}, \mathcal{L}(m^\varepsilon(\cdot,t))\right) & (t,x)\in(0,T)\times\mathbb{T}^n \\ m_t^\varepsilon - \varepsilon\Delta m^\varepsilon - \mathrm{div}(m^\varepsilon\nabla u^\varepsilon) = 0 & (t,x)\in(0,T)\times\mathbb{T}^n, \end{cases}$$

where $w^\varepsilon := \mathcal{L}m^\varepsilon$ is the periodic function such that $-\Delta w^\varepsilon = m^\varepsilon - 1$ on $\mathbb{T}^n$, with $w^\varepsilon(0) = 0$.
We add the ansatz that also w^ε satisfies the asymptotic expansion

$$w^\varepsilon(x,t) = w^0(x,t) + \varepsilon^2 w\left(\frac{x}{\varepsilon}\right),$$

where $-\Delta w^0 = m^0 - 1$, where w is a periodic function with zero average such that $\Delta w(y) = m^0(x)(m(y) - 1)$. Also in this case the cell system decouples.

Properties of the effective operators

Lemma

$$\lim_{|P|\to+\infty} \frac{\bar{H}(P,\alpha)}{|P|^2} = \frac{1}{2} \qquad \lim_{|P|\to+\infty} \frac{|\bar{b}(P,\alpha) - P|}{|P|} = 0 \qquad \text{unif. in } \alpha.$$

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Proof: For $P \in \mathbb{R}^n$, $\alpha > 0$ fixed, let (u_P, m_P) be the solution to

$$\begin{cases} (i) & -\Delta u_P + \frac{1}{2}|\nabla u_P + P|^2 - V(y, \alpha m_P) = \overline{H}(P,\alpha) \qquad \int_{\mathbb{T}^n} u_P = 0 \\ (ii) & -\Delta m_P - \mathrm{div}\,(m_P(\nabla u_P + P)) = 0 \qquad \int_{\mathbb{T}^n} m_P = 1. \end{cases}$$

- multiply (i) by $m_P - 1$, (ii) by u_P, integrate and substract: $\int_{\mathbb{T}^N} \frac{|\nabla u_P|^2}{2}(m_P+1) + V(y,\alpha m_P)(m_P-1)dy = 0$;
- by monotonicity: $V(y,\alpha m_P)(m_P-1) \geq V(y,\alpha)(m_P-1)$;
- $|\nabla u_P|/|P|, \sqrt{m_P}|\nabla u_P|/|P| \to 0$ in L^2
- we integrate (i) on the torus, divide by $|P|^2$ and send $|P| \to +\infty$
- $|\bar{b} - P| \leq \int_{\mathbb{T}^n} |\nabla u_P| m_P \leq \left(\int_{\mathbb{T}^n} |\nabla u_P|^2 m_P\right)^{1/2} \left(\int_{\mathbb{T}^n} m_P\right)^{1/2}$.

Theorem

The following formulas hold:

i) $$\nabla_P \bar{H}(P,\alpha) = \bar{b}(P,\alpha) - \int_{\mathbb{T}^n} V_m(y,\alpha m)\alpha m(\nabla_P m)dy$$

ii) $$\partial_\alpha \bar{H}(P,\alpha) = -\int_{\mathbb{T}^n} \left[V_m(y,\alpha m)(m+\alpha\partial_\alpha m)^2 + \alpha m|\nabla \partial_\alpha m|^2 \right] dy$$

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Heuristic derivation: **Variational characterization of $\bar{H}$.**

$$E_{P,\alpha}(u,m) = \int_{\mathbb{T}^n} \frac{|\nabla u + P|^2}{2} m + (\nabla u + P)\cdot\nabla m - \Phi_\alpha(y,m)dy$$

where $\Phi_\alpha(y,m) = \int_0^m V(y,\alpha s)ds$. We have

$$\partial_m E_{P,\alpha}(u,m) = 0 = \partial_u E_{P,\alpha}(u,m) \iff (u,m) \text{ solves the cell problem}$$

$$\bar{H}(P,\alpha) = E_{P,\alpha}(u,m) + \int_{\mathbb{T}^n} \Phi_\alpha(y,m) - V(y,\alpha m)m.$$

Proof of (i).

- Let (u, m) and (u_δ, m_δ) be the sol. to cell problem with (P, α) and resp. with $(P + \delta e_i, \alpha)$. Let $w_\delta = (u_\delta - u)/\delta$, $n_\delta = (m_\delta - m)/\delta$.
- a priori L^2 bounds on ∇w_δ, n_δ
- w_δ, n_δ are weakly converging (in H^1 and L^2) to u_i, m_i, solution to

$$\begin{cases} -\Delta u_i + \nabla u_i \cdot (\nabla u + P) + (\nabla u + P) \cdot e_i - V_m(y, \alpha m)\alpha m_i = c_i \\ -\Delta m_i - \operatorname{div}\big((P + \nabla u)m_i\big) = \operatorname{div}(m(\nabla u_i + e_i)) \\ \int_{\mathbb{T}^n} m_i = \int_{\mathbb{T}^n} u_i = 0. \end{cases}$$

- $c_i = \bar{b}(P, \alpha) \cdot e_i - \int_{\mathbb{T}^n} V_m(y, \alpha m)\alpha m m_i dy$

Is the limit system always a MFG?

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Answer

NO.

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Heuristic explanation: MFG with local coupling can be interpreted as an optimality condition between two problems in duality. One is

$$\min_{(\mu,w)\in K_{1_\varepsilon}} \int_0^T \int_{\mathbb{T}^n} \frac{|w(t,x)|^2}{\mu(t,x)} + F(x/\varepsilon, \mu(t,x))dxdt + \int_{\mathbb{T}^n} u_0(x)\mu(T,x)dx$$

where $F(y,s) := \int_0^s V(y,r)dr$ while K_{1_ε} is formed by the couples $(\mu, w) \in L^1 \times (L^1)^n$, $\mu \geq 0$ *a.e.*, $\int \mu(t,x)dx = 1$, $\mu(0,x) = m_0(x)$ and

$$\mu_t - \varepsilon\Delta\mu + \mathrm{div}(w) = 0.$$

[Cardaliaguet, Graber, Porretta, Tonon]: there exists a unique minimizer $(\bar{\mu}, \bar{w})$; moreover: $\bar{\mu} = m^\varepsilon$. The “dual” problem “gives” u^ε.

An explicit counter-example. Assume $n = 1$ and

$$V(y, m) = v(y) + m$$

(not in the same assumptions, but we can extend all the results).
The formula reads:

$$\frac{\partial \bar{H}(P, \alpha)}{\partial P} = \bar{b}(P, \alpha) - \frac{\alpha}{2} \frac{\partial \|m\|_2^2}{\partial P}$$

Lemma

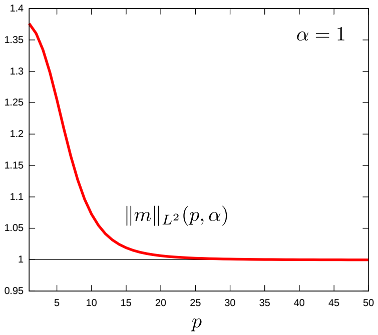
$$\frac{\partial \|m\|_2^2}{\partial P} \neq 0.$$

Proof. By definition $\|m\|_1 = 1$ and $m \not\equiv 1$. By Jensen:

$$\|m\|_2^2 > 1 = \|m\|_1^2.$$

It is sufficient to prove that $m \to 1$ strongly in L^2 as $|P| \to +\infty$.

- $\|m\|_2$ is uniformly bounded in P (reasoning as above)
- $m \to 1$ weakly in L^2 (using the equation)
- $\sqrt{m}$ is uniformly bounded in P in H^1 (multipy the equation for m by $\log m$ and integrate)
- H^1 compactly embedded in C^0; hence $m \to 1$ in L^2.



(by courtesy of Cacace and Camilli)

here $\alpha = 1$, $v(y) = 50(\sin(2\pi y) + \cos(4\pi y))$.

Convergence result

Fix $P \in \mathbb{R}^n$ and consider the mean field game system

$$\begin{cases} -u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + \frac{1}{2}|\nabla u^\varepsilon|^2 = V\left(\frac{x}{\varepsilon}, m^\varepsilon\right) & (t,x) \in (0,T)\times\mathbb{R}^n \\ m_t^\varepsilon - \varepsilon\Delta m^\varepsilon - \mathrm{div}(m^\varepsilon \nabla u^\varepsilon) = 0 & (t,x) \in (0,T)\times\mathbb{R}^n \\ u^\varepsilon(x,T) = P\cdot x \qquad m^\varepsilon(x,0) \equiv 1 & x \in \mathbb{R}^n. \end{cases} \tag{1}$$

The data $P \cdot x$ is not periodic: we rotate the reference system.

The limit system reads

$$\begin{cases} -u_t + \bar{H}(\nabla u, m) = 0 & u(x,T) = P\cdot x \\ m_t - \mathrm{div}(\bar{b}(\nabla u, m)m) = 0 & m(x,0) = 1; \end{cases}$$

it has the trivial solution

$$u^0(x,t) = P\cdot x + (t-T)\bar{H}(P,1) \qquad m^0(x,t) \equiv 1.$$

Theorem

For every compact Q in $\mathbb{R}^n$,

- $u^\varepsilon \to P \cdot x + (t - T)\bar{H}(P, 1)$ in $L^2([0, T] \times Q)$
- $m^\varepsilon \to 1$ weakly in $L^p([0, T] \times Q)$ for $1 \leq p < (n + 2)/n$ if $n \geq 3$ and $p < 2$ if $n = 2$.

Main tools: energy estimates

We borrow main ideas from Cardaliaguet-Lasry-Lions-Porretta for the long time behavior.

Sketch of proof

Define

$$\begin{cases} v^\varepsilon(y,t) = \frac{1}{\varepsilon} u^\varepsilon(\varepsilon y, t) - P \cdot y - \frac{1}{\varepsilon}(t-T)\bar{H}(P,1) - u(y) \\ n^\varepsilon(y,t) = m^\varepsilon(\varepsilon y, t) - m(y) \end{cases}$$

where (u, m) is the solution to the cell problem with $(P, 1)$.

Step 1. For every $0 \leq t_1 \leq t_2 \leq T$, there holds

$$-\varepsilon \left[\int_{\mathbb{T}^n} v^\varepsilon n^\varepsilon \right] \Big|_{t_1}^{t_2} =$$

$$= \int_{t_1}^{t_2} \int_{\mathbb{T}^n} \frac{2m + n^\varepsilon}{2} |\nabla v^\varepsilon|^2 + [V(y, m+n^\varepsilon) - V(y,m)]\, n^\varepsilon dy.$$

Proof. We cross test the equations fulfilled by $(v^\varepsilon, n^\varepsilon)$.

Step 2. There holds:

$$\int_0^T \int_{\mathbb{T}^n} \frac{2m + n^\varepsilon}{2} |\nabla v^\varepsilon|^2 + [V(y, m + n^\varepsilon) - V(y, m)]\, n^\varepsilon dy \le \varepsilon C.$$

Proof. We introduce the energy

$$E(v, n) =$$

$$= \int_{\mathbb{T}^n} (n + m)[(\frac{|\nabla v|^2}{2} + \nabla v \cdot (P + \nabla u)) + \nabla v \cdot \nabla(n + m) - \Phi^1(y, n)]dy$$

where $\Phi^1(y, n) = \int_0^n (V(y, s + m) - V(y, m))ds$. We use Step-1 and

$$\frac{dE}{dt}(v^\varepsilon(\cdot, t), n^\varepsilon(\cdot, t)) = 0.$$

Step 3. There holds

$$(i) \qquad \lim_{\varepsilon\to 0} \nabla v^\varepsilon = 0 \qquad \text{in } L^2((0,T)\times\mathbb{T}^n)$$

$$(ii) \qquad \lim_{\varepsilon\to 0} n^\varepsilon = 0 \qquad \text{in } L^p((0,T)\times\mathbb{T}^n).$$

Proof of (i). Use Step-2 and $2m + n^\varepsilon \geq c > 0$.

Proof of (ii). We use

- Monotonicity of V and Step-2 entail: $\lim_{\varepsilon\to 0} n^\varepsilon = 0$ in L^1;
- $m^\varepsilon(\varepsilon x, t)$ are uniformly bounded in $L^{\bar p}$ for any $\bar p \leq (n+2)/n$ if $n \geq 3$, for any $\bar p < 2$ if $n = 2$.
- $m^\varepsilon(\varepsilon x, t)$ weakly converge to m^1 in $L^{\bar p}$ for any $\bar p \leq (n+2)/n$ if $n \geq 3$, for any $\bar p < 2$ if $n = 2$.

Step 4. Go back to $(u^\varepsilon, m^\varepsilon)$ rescaling the results in Step-3.

Perspectives

- Other properties of the effective operators;
  - does $\bar{b}$ depend on α?
  - What about monotonicity of the limit problem?
  - When the MFG structure is preserved? This happens for $V = v(y) + \log m$...work in progress.
- Convergence for general system?
- Rate of convergence?

Thank you for your attention!