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SUMMARY

I INTRODUCTION

II NOT DISCUSSED HERE

III FN_L ELLIPTIC CASE

IV MAXIMAL SOLUTIONS

V SCENES FROM OUR NEXT EPISODE

(TO BE CONT'D TAKIS PT II)

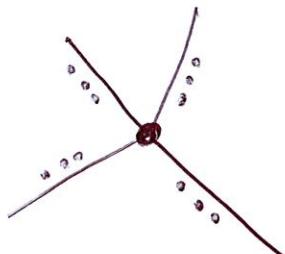


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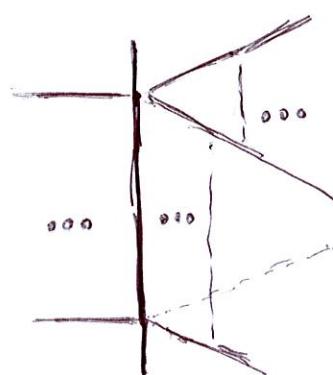
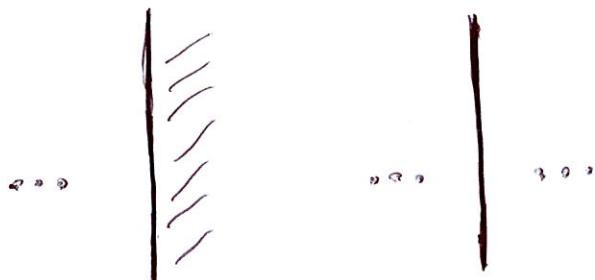
I INTRODUCTION

- PBS CONSIDERED HERE

1D



D ≥ 2



... = VISC. SOL. Eq (1st order or 2nd order ...)



GOAL: "GLOBAL" THEORY ($2^{nd}, 1^{st}, 2^{nd} \rightarrow 1^{st}$) for general Hamiltonians and multi-D eqs.

HJ Eq = HJ Eq (+ 2nd order deg. at "0" in — dir.)

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• WHY AND WHO : HUGE LITERATURE

SÖNER 86 ; DUPUIS 92

GARAVELLO - SORAVIA 04, 06 ; SORAVIA 06, DEZER - SORAVIA 10

DECKENICK - ELLIOTT 04 , COCLITE - RISEBRO 07

BRESSAN - YONG 07

BLANC 97, 01

GIGA - GÖRKE - RYBKA 11

BARLES - BRIANI - CHASSEIGNE 13, 14 } ←
 BARLES - CHASSEIGNE 15 }

CAMILLI - SICONOLFI 05

CAMILLI - MARCHI 14

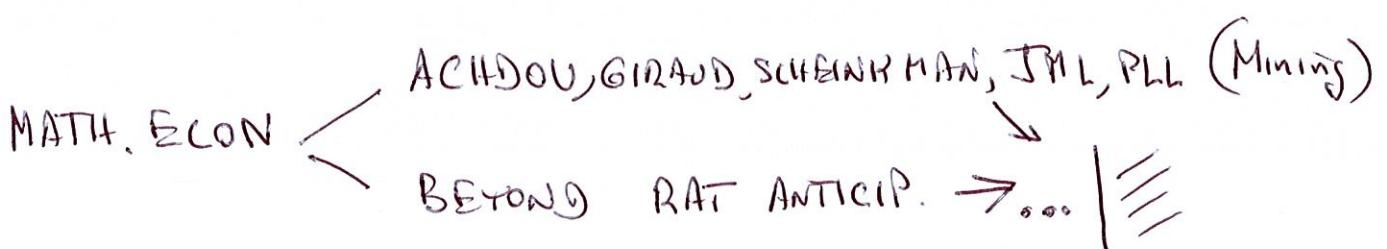
ACHDOU - CAMILLI - CUTRI - TCHOU 13

CAMILLI - SCHNEIDER 13

RAD - ZIDANI 13

RAD - SICONOLFI - ZIDANI 15

IMBERT - MONNEAU 15 } ←
 IMBERT - NGUYEN 16 }



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IT NOT DISCUSSED HERE

→ LAST LECTURES AT CdF

PBS COMING FROM ECON.

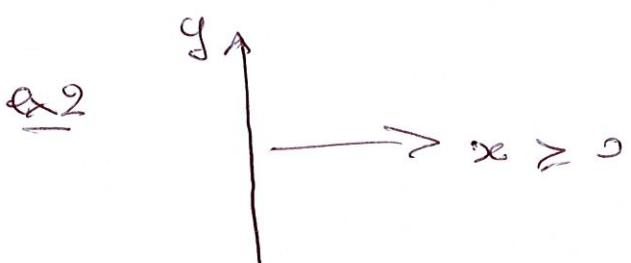
ex 1 $x \geq 0$

$$x \cdot u_x^2 + u = f(x) \quad (H(x, u_x) + u = 0)$$

$$\text{in general } u(0) < f(0) \quad | \quad \begin{aligned} & H_u \approx x^{(\alpha-1)\beta} \quad \alpha > \beta \\ & \text{at } \infty \\ & \text{not necc. convex} \end{aligned}$$

state constraints cond at 0 work: $\exists!$ --- limit of trac. Hmbr.

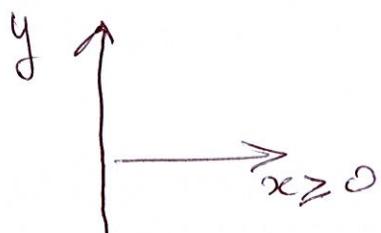
(degenerate at 0 2nd order terms OK)



$$x H_1 + H_2(x, y) u_y \\ \approx u_x^2 + H_2(u_y) \text{ ex.}$$

(not nec. convex, "behavior at 0", 2nd order terms deg in x at x=0)

ex. 3



$$x > 0 \quad H_1 \quad (+\text{2nd order terms as above})$$

$$\text{at } x=0 \quad H_2(y, u_y) \quad (+\text{2nd order terms})$$

general (H_1, H_2) : $\exists!$ sol. s.t. $\text{at } x=0$

$H_1 \rightarrow \infty$ at ∞ , $u(0, y)$. H_2 -substitution in y

• Supersolution ($x \geq 0, y$) $\max(H_1, H_2)$

→ ALSO NOT COVERED HERE

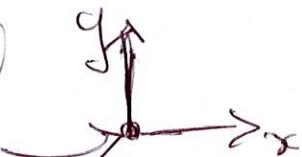
WORSE ISOLATED SINGULARITIES



$$\text{ex. } |x|^\alpha(\omega) \quad |\nabla u|^2 + u = f(x) \\ \alpha > 0$$

(CONVEX HAMILT... BUT IN PROGRESS)

(needed for mining model



AND MATERIAL DISCUSSED HERE AND IN

TAKIS LECTURE \approx HALF OF ME

CDF COURSE NEXT FALL

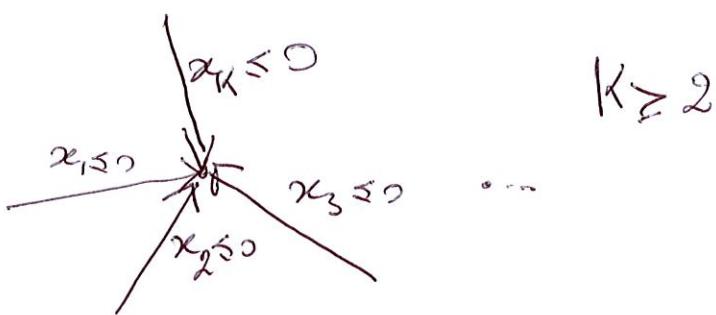
(other half: beyond classical stochastic control

control of V.-MkR processes, control and Bayesian inference,
control of conditioned processes)

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III FINL ELLIPTIC CASE (1D FOR PARABOLIC...)

1D JUNCTIONS



$$u = u_1(x_1), = u_2(x_2), = u_3(x_3) \quad \text{CONT. AT } 0$$

For $x_i < 0$, $F_i(x_i, \frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i^2}) = 0$

ex. $\frac{\partial F_i}{\partial a} \leq -v < 0$

strictly elliptic (F_i Lip in all variables...)

Let $G(p_1, \dots, p_K, z)$ \nearrow in all variables. (Lip)

$$\frac{\partial G}{\partial p_i} \geq v > 0.$$

Prop. 3! $u \in C^1(x_i < 0)$ s.t. at 0 (see $C^{1,1}$)

$$G\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_K}, u\right) = 0$$

"Proof" u substitution, or superlution (C^1), $\max u$ at $0 > 0$

$$\frac{\partial u}{\partial x_i}(0) \geq \frac{\partial v}{\partial x_i}(0), \text{ BC} \Rightarrow \text{ equality class rel max p. !}$$

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RKI : FOR LINEAR ELLIPTIC OPERATORS AND ALINBAR
BDRY COND.

$$(*) \quad \lambda_0 u + \sum_{i=1}^k \lambda_i \frac{\partial u}{\partial x_i} = 0 \quad \lambda_i > 0, \lambda_0 \geq 0$$

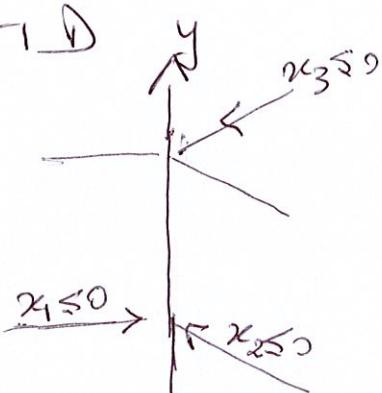
(dn. form op., no 1st order terms $\Leftrightarrow \lambda_0 > 0$, the whole system is conservative IFF

$$(\lambda_1, \dots, \lambda_k) = \text{Cst } (a_1(0), \dots, a_k(0))$$

where $a_i(x)$ is the diffusion coefficient)

($\int \frac{\partial^2}{\partial x^2}$, Brownian motions (*) is related to "splitting probabilities" + λ_0 is related to "walking" time at ∂)

RK2 : MULTID



SAME WITH

$$G\left(\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_k}, \frac{\partial^2 u}{\partial y^2}, \frac{\partial^2 u}{\partial y \partial x_i}, u, y\right) = 0$$

$\uparrow, \dots, \uparrow, \downarrow, \times, \uparrow, \times$

IV MAXIMAL SOLUTIONS

① 1D $K \geq 2$ (\mathbb{R}^K $x_1 \leq 0 \quad x_2 \leq 0 \quad x_3 \leq 0$)

$$\text{For } x_i < 0 \quad H_i(x, \frac{\partial u}{\partial x_i}) + u = 0$$

Well-known "continuum" of solutions parametrized by $u(0)$

$H_i \rightarrow \infty$ as $\varphi_i \rightarrow \infty$ ("needed" at 0, extensions are clear)

not necessarily convex.

Questions: i) limit ($K \geq 2$) of $\varepsilon u_{x_i x_i}$ as $\varepsilon \rightarrow 0$
and, for instance, $\sum_{i=1}^K u_{x_i} = 0$? , ii) limit ($R=2$)

of smoothed Hamiltonian

$$\frac{H_1}{x \leq 0} \quad \frac{H_2}{x \geq 0} \quad \left(= -x_2 \quad p_2 \rightarrow p_2 \right)$$

iii) maximal solution ?, iv) "flattening" limit, v)

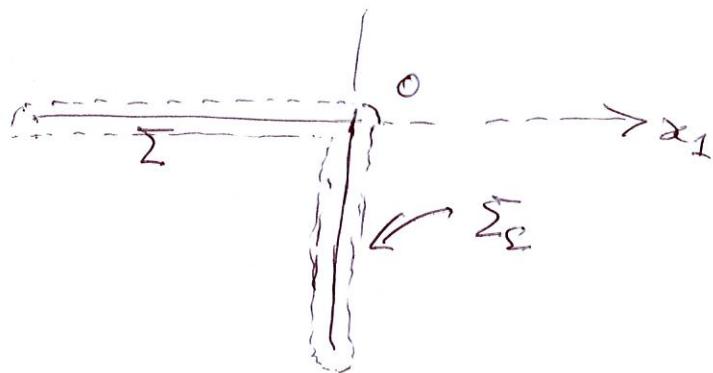
parametrization of solutions, vi) multidim.

(time dependent similar with a few more technical details ...)

② FATTENING AND THE SUPERPOSITION COND. ARE

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$H \geq 2$



$\Sigma_\varepsilon = \varepsilon\text{-neigh. of } \Sigma \text{ in } \mathbb{R}^K \rightarrow H: \mathbb{R}^{2K} \rightarrow \mathbb{R}$
 $H \rightarrow +\infty \text{ at } \infty$

$$\left\{ \begin{array}{l} H(x_1, x_2, \frac{\partial u_\varepsilon}{\partial x_1}, \frac{\partial u_\varepsilon}{\partial x_2}) + u_\varepsilon = 0 \text{ in } \Sigma_\varepsilon \\ \text{S.C. bdy cond. on } \Sigma_\varepsilon \end{array} \right.$$

$u_\varepsilon(u_{\varepsilon_n}) \rightarrow u$ (Lip bds) uniformly

Prop.: i) u is viscosity sol of

$$H_1(x_1, \frac{\partial u}{\partial x_1}) + u = 0 \text{ on } \{x_1 < 0, x_2 = 0\}$$

$$H_2(x_2, \frac{\partial u}{\partial x_2}) + u = 0 \text{ on } \{x_2 < 0, x_1 = 0\}$$

ii) and, for all $\varphi \in C^1(\mathbb{R}^K)$, if $u - \varphi$ has a local min on Σ at 0 then

$$H(0, \nabla \varphi) + u \geq 0$$

$$(H_\pm(x_1, p_1) = \min_{p_2} H(x_1, 0, p_1, p_2), H_2(x_2, p_2) = \min_{p_1} H(0, x_2, p_1, p_2))$$

Proof of ii) (ii) is true \Rightarrow

* $u(x_1, 0) - \varphi(x_1)$ has a local min. at $\bar{x}_1 > 0$.

... $u_\varepsilon(x_1, x_2) - \varphi(x_1) - p_2 x_2$ has a local min. at $\bar{x}_1^{\varepsilon} \rightarrow \bar{x}_1$,
 $\bar{x}_2^{\varepsilon} \rightarrow 0$

$$H(\bar{x}_1, 0, \varphi'(\bar{x}_1), p_2) + u(\bar{x}_1) \geq 0$$

$$\nabla p_2! \Rightarrow H_1(\bar{x}_1, \varphi'(\bar{x}_1) + u(\bar{x}_1)) \geq 0$$

$$\begin{aligned} *) & H_1(x_1, \cancel{\frac{\partial u}{\partial x_1}(x_1, x_2)} + \frac{\partial u^\varepsilon}{\partial x_1}(x_1, x_2)) + u_\varepsilon \\ & \leq H(x_1, 0, \frac{\partial u^\varepsilon}{\partial x_1}(x_1, x_2), \frac{\partial u^\varepsilon}{\partial x_2}(x_1, x_2)) + u_\varepsilon \\ & \leq H + u_\varepsilon + o(\varepsilon) \leq o(\varepsilon) \end{aligned}$$

Cor. If $H = \max(H_1(x_1, p_1), H_2(x_2, p_2))$, \exists a sol of

$$H_1 + u = 0 \text{ on } \{x_1 < 0\}, \quad H_2 + u = 0 \text{ on } \{x_2 < 0\}.$$

s.t. $\forall \varphi \in C^1(\mathbb{R}^K)$

0 loc min of $u - \varphi$ on $\Sigma \Rightarrow$

$$u(0) + \max(H_1(0, \frac{\partial \varphi}{\partial x_1}), H_2(0, \frac{\partial \varphi}{\partial x_2})) \geq 0$$

(Supersolution at 0 in \mathbb{R}^K)

③ MAXIMAL SOLUTION

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THM: $H_1, H_2, \dots \rightarrow \infty$ as $|P| \rightarrow \infty$

i) $\exists! u$ sl.s.f. $u + H_i = 0$ on $\{x_i < 0\}$ which is a superdulation of $\Omega \subset \mathbb{R}^K$

ii) If v satisfies $v + H_i \leq 0$ on $\{x_i < 0\}$ then $v \leq u$

iii) $u(\omega) = \min_K u_i^{sc}(\omega)$

where u_i^{sc} is the slcte. constraint solution of $u + H_i = 0$

for $x_i < 0$

Proof: existence by Perron (Fatou's lemma) works only if

$$\min_{P_1} H_1 = \min_{P_2} H_2 \quad \max \{v(x) / H_i \leq 0 \text{ on } \{x_i < 0\}\} \text{ v cont. at } 0$$

uniqueness v subdulation, u such a solution is unique?

Soner's proof on each $x_i < 0$

$$\max_{x'_i; x_i < 0} v(x'_i) - u(x_i) - \frac{1}{2\varepsilon} (x'_i - x_i + \delta)^2$$

$$\cdot \bar{x}'_i < 0, \bar{x}'_i \leq 0 \quad (\delta \Leftrightarrow \varepsilon, \text{lip: } \delta = M\sqrt{\varepsilon})$$

- If for some i $\bar{x}_i < 0$ over!

- $\forall i \bar{x}_i = 0$ then $u + \sum_i \frac{1}{2\varepsilon} (\bar{x}'_i - x_i + \delta)^2$ has a min at 0.

Hence $u(0) + \max_{1 \leq i \leq K} H_i(0, \frac{\bar{x}_i' + \delta}{\varepsilon}) \geq 0$

while $v(\bar{x}_i') + H_i(\bar{x}_i', \frac{\bar{x}_i' + \delta}{\varepsilon}) \leq 0 \quad \dots$

RB $K=2$ $\overline{H = H_1 \cup H_2} = H_2(-x, -r)$

superd at 0 on Σ means if

$$u(x) \geq u(0) + px^- + qx^+ + o(|x|)$$

then $u(0) + \max(H_1(0, p), H_2(0, q)) \geq 0$

(classical formulation $p=q$ not enough ...)

(*) $H_+(0, p) = \inf_{x > 0} H_+(x, p) \Rightarrow \begin{cases} \text{not true if } \Sigma \\ \text{nonconvex} \end{cases}$

classical viscosity solutions are maximal solutions at 0

(a.e. solution are viscosity sub-solutions!)

**) Smooth the discontinuity: H_ε continuous connecting
 $H_- \rightarrow H_\varepsilon \text{ for } x < 0, \rightarrow H_+ \text{ for } x > 0$
 $H_- + H_+ = 0 \text{ on } \mathbb{R}$

then $u_\varepsilon \rightarrow u$ max sol.

(u_ε is a max sol at 0 + stability of max sol. at 0)

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$$(5) -\varepsilon u_{xx}^{\varepsilon} \quad (H_i \text{ convex} \dots)$$

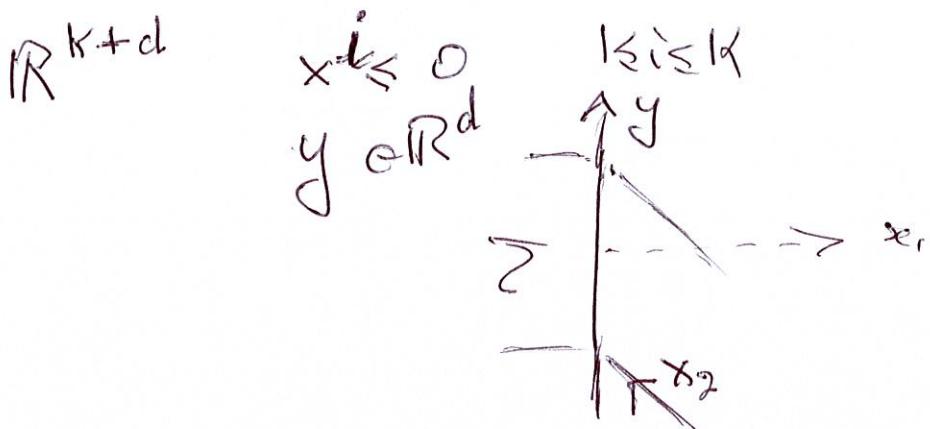
$\mu_{\varepsilon} \xrightarrow{\varepsilon} \mu$ max s.t. at 0 if $\sum_i p_i \leq 0$

where $p_i = \arg \min H_i(0, p_i)$

(IFF in Takao's lecture $\mu_{\varepsilon} \xrightarrow{\varepsilon} \mu$ + nonconvex...)

(6) Multi-D

combination of previous remarks and CdF lectures



$$H_i(x_i, y) \frac{\partial u}{\partial x_i}, \nabla_y u) + u = 0 \quad \text{on } \{x_i < 0\}$$

$$\text{at } (0, y_0) \dots \max_{1 \leq i \leq K} H_i(0, y_0) \frac{\partial \varphi}{\partial x_i}, \nabla_y \varphi) + u(0) \geq 0$$

(Sinf condition and "some proof" \Rightarrow some results as above!)

(7) An extension

same as before + " $F(y, \nabla u) + u = 0$ " for $x=0$
 $H_\varepsilon, F \rightarrow \infty$ at y

same results for "precise" maximal solutions:

- $H_i + u = 0$ on $\{x_i < 0\} \times \mathbb{R}^d$
- $u(0, y)$ solution on $\{0\} \times \mathbb{R}^d$ $F(y, \nabla u) + u = 0$
- $u(0, y)$ supersolution at $\{0\} \times \mathbb{R}^d$ (with respect to local min in \sum all variables) of
 $u + \max(F(y, \nabla u), \max_{1 \leq i \leq K} \left\{ H_i(0, y, \frac{\partial u}{\partial x_i}, \nabla u) \right\}) \geq 0$

(8) "Distance functions"

may be difficult to define: one way smooth $H_\varepsilon \rightarrow L_\varepsilon(x, y)$

$L_\varepsilon \rightarrow L$ maximal sol. (maximal "distance")

viscosity sol's with L-hat flows are maximal solutions ...

V SCENES FROM OUR NEXT EPISODE

(preview of Takis lecture Pt II)

- Imbert-Monneau notion and uniqueness prf,

- $-\varepsilon u_{xx}$ Hamilton : $u_\varepsilon \rightarrow u$

(catalogue of solutions)

- identification of the limit

Lemma: $H(x, u_x) + u = 0$ for $x < 0$

u cont at 0

$$\Rightarrow \lim_{x \rightarrow 0^-} \frac{u(x) - u(0)}{x} \leq p \quad \text{or } u \text{ S.C. sd.}$$

$$p = \inf \{ z / \forall p \geq z \quad H(0, p) \geq H(0, z) \}.$$

Ex. "strictly" convex ... $u'(0) < p_{\min} = \arg \min H(0, p)$

or u S.C. sd.

(and mult.D extensions -->

Ex. $|u'|_{\infty} = 1$

