# Weak solutions of mean field games systems

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*HJ 2016: "Hamilton-Jacobi equations, new trends and applications"*

Rennes, 31/05/2016

# Outlines of the talk

- Mean Field Games systems lead naturally to work with unbounded solutions to Hamilton-Jacobi equations.

  - Even for viscous HJ, this leads to non trivial uniqueness problems.

  - The role of the (adjoint) Fokker-Planck equation is crucial in this study

- Second order case
  $\rightarrow$ Fokker-Planck with $L^2$-drift & weak solutions to MFG systems

- Vanishing viscosity and first order case
  $\rightarrow$ new estimates for HJ with data in Lebesgue spaces

# Mean field games systems

Mean field games theory ([Lasry-Lions], [Huang-Caines-Malhamé]) leads to coupled systems of PDEs. Simplest case (in a time horizon $T$):

$$\begin{cases} (1) \qquad -u_t - \Delta u + H(t,x,Du) = F(t,x,m) & \text{in } (0,T)\times\Omega \\ (2) \qquad m_t - \Delta m - \mathrm{div}(m\, H_p(t,x,Du)) = 0 & \text{in } (0,T)\times\Omega\,, \end{cases}$$

where $H_p$ stands for $\frac{\partial H(t,x,p)}{\partial p}$.

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where $H_p$ stands for $\frac{\partial H(t,x,p)}{\partial p}$.

- (1) is the Bellman equation for the agents' value function $u$.
- (2) is the Kolmogorov-Fokker-Planck equation for the distribution of agents. $m(t)$ is the probability density of the state of players at time $t$.

Typically: $p \mapsto H(t,x,p)$ is convex.

Model ex: $H \simeq \gamma(t,x)|Du|^q$.

# Macroscopic derivation

**1. Individual optimization:** each agent controls a $N$-d Brownian motion

$$dX_s = \beta_s ds + \sqrt{2}\, dB_s\,, \quad X_t = x$$

in order to minimize, among controls $\beta_s$, the cost

$$J_{t,x}(\beta) := \mathbb{E}_{t,x}\left\{\int_t^T [L(X_s,\beta_s) + F(X_s, m(s))]ds + G(X_T, m(T))\right\}$$

where $m(t)$ is a probability measure (the *supposed* law of $X_t$).

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**DPP:** The value function $u(t,x) := \inf_\beta J_{t,x}(\beta)$ solves the Bellman eq:

$$-u_t - \Delta u + H(x, Du) = F(x, m(t))$$

where $H$ is the Legendre transform $H = \sup_\beta[-\beta\cdot p - L(x,\beta)]$. This gives

- the best value $\inf_\beta J(\beta) = \int u(x,0)dm_0(x)$, $\quad m_0 = \mathcal{L}(X_0)$.
- the optimal control through the feedback law:

$$\beta^*_t = -H_p(X_t, Du(t,X_t))\,, \qquad H_p := \frac{\partial H(x,p)}{\partial p}$$

2. **Evolution of the collective state**: Given a drift-diffusion process

$$dX_t = b(t, X_t)dt + \sqrt{2}dB_t$$

the probability measure $m(t)$ (distribution law of $X_t$) satisfies

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The system is at equilibrium if the optimal feedback used by the agents $b_t^* = -H_p(\cdot, Du(\cdot))$ produces the distribution measure $m(t)$ which was assumed in the optimization process.

$$\to \begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m) \\ m_t - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 \end{cases},$$

This is the Mean Field Games system (with horizon $T$):

$$\begin{cases} (1) & -u_t - \Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ (2) & m_t - \Delta m - \operatorname{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \end{cases}$$

usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents)

-$u(T) = G(x, m(T))$ (final pay-off)

$+$ boundary conditions (here for simplicity assume periodic b.c.)

Main novelties are:

- the backward-forward structure.
- the interaction in the strategy process: the coupling $F(x, m)$

Rmk 1: This is not the most general structure.
Cost criterion $L(X_t, \alpha_t, m(t)) \to H(x, m, Du)$.

Rmk 2: The special structure $H = H(Du) - F(m)$ gives to the system a variational structure $\to$ optimality system (so-called *mean field control systems*)

**Key assumption** [Lasry-Lions]:

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However, the existence of smooth solutions is mostly unknown:

Minimization of the cost $\mathbb{E}\left[\int_0^T F(m(t))dt\right] < \infty$

$\Rightarrow$ Estimates of the system only give $\iint F(m)m \leq C$

Typical example: $F(m) \simeq m^\gamma$:

$$\rightarrow \; m^{\gamma+1} \in L^1 \quad \Rightarrow \quad F(m) \in L^{\frac{\gamma+1}{\gamma}}$$

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Partial results:

- the existence of smooth solutions is known in few cases:

  (i) if $m \mapsto F(x,m)$ or $p \mapsto H(x,p)$ have a mild growth ([Lasry-Lions], [Gomes-Pimentel-Sanchez Morgado])

  (iii) in the homogeneous quadratic case $H(x,p) = |p|^2$, solutions are smooth *for every* $F(x,m) \geq 0$ ([Cardaliaguet-Lasry-Lions-P.])

Pb: build a complete theory of weak solutions (existence, uniqueness, stability...).

Motivations & applications:

- Convergence of numerical schemes

- Existence and uniqueness for the planning problem (prescribed initial and final densities $m(0)$ and $m(T)$):

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m) \,, \\ m_t - \Delta m - \text{ div } (m \, H_p(x, \nabla u)) = 0 \,, \\ m(0) = m_0 \,, \;\; m(T) = m_1 \end{cases}$$

  Here, no condition is assumed on $u$ at time $T$.

  This is an optimal transport model for the distribution law $m$ of the stochastic flow.

Main problems for a weak theory:

**1.** The typical setting for well-posedness of the Fokker-Planck

$$(FP) \qquad m_t - \Delta m + \operatorname{div}(m\, b) = 0 \qquad (t,x) \in (0,T)\times\Omega\,, \quad \Omega \subset \mathbb{R}^N$$

requires a high integrability for the drift $b$:

$$b \in L^\infty(0,T; L^N(\Omega))\,, \qquad \text{o} \quad b \in L^{N+2}((0,T)\times\Omega)$$

(or any interpolation between the two conditions, cfr. [Aronson-Serrin], [Ladysenskaya-Solonnikov-Uraltseva])

# Weak theory for MFG systems

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Pb. MFGames: $b = H_p(x, Du) \simeq |Du|^{q-1}$ is in the right class only if $q$ is small or $Du$ highly integrable

**2.** Uniqueness may fail for unbounded solutions of HJB:

$$\exists \;\; u \in L^2(0,T;H^1_0)\,,\;\; u \neq 0 \text{ sol. of } \quad \begin{cases} u_t - \Delta u + |Du|^2 = 0 \\ u(0) = 0 \end{cases}$$

Counterexamples are constructed as $u = \log(1+v)$, $v$ solutions to

$$\begin{cases} v_t - \Delta v = \chi \\ v(0) = 0 \end{cases}$$

provided $\chi$ is a concentrated measure (ex. $\chi = \delta_{x_0}$)

Back to MFG system:

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m)\,, \\ m_t - \Delta m - \ \text{div}\ (m\, H_p(x, Du)) = 0\,, \end{cases}$$

Summary:

(i) HJB has no uniqueness of weak solutions

(ii) The drift in FP might not have the right summability

Desperate situation ?......

Back to MFG system:

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Desperate situation ?...... But MFG solutions satisfy an extra estimate:

$$m\, L(x, H_p(x, Du)) \in L^1(Q_T) \tag{1}$$

which comes from optimization:

$$\int_0^T \int_\Omega L(x, H_p(x, Du))\, m \simeq \mathbb{E}\left[\int_0^T L(X_t, H_p(X_t, Du(t, X_t)))dt\right] < \infty$$

Ex: model case: $L, H$ with quadratic growth

$$(1) \Rightarrow m\,|Du|^2 \in L^1\,, \quad \leftrightarrow \quad m\,|H_p(x, Du)|^2 \in L^1$$

# A new approach to Fokker-Planck equation

Key point: we can consider solutions of Fokker-Planck

$$m_t - \Delta m - \text{ div } (bm) = 0$$

such that $m \geq 0, \quad m|b|^2 \in L^1$

In this framework, we can prove:

1. Weak (=distributional) solutions of (FP) are unique in this class
2. Weak solutions are renormalized solutions;

   (in the sense of [Di Perna-Lions], extended to second order, cfr. [Lions-Murat])

   Moreover, those solutions can be regularized and obtained as limit of smooth solutions.

Rmk: The importance of the class $\{m : b \in L^2(m)\}$ was also stressed in [Bogachev-Da Prato-Röckner '11], [Bogachev-Krylov-Röckner]

The typical statement is the following (adapted to Dirichlet, Neumann, or to entire space $\mathbb{R}^N$ under suitable modifications)

**Theorem (P., ARMA 2015)**

*Let $b \in L^2(Q_T)^N$ and $m_0 \in L^1$. Then the problem*

$$\begin{cases} m_t - \Delta m - \mathrm{div}(m\, b) = 0\,, & \text{in } (0,T)\times\Omega, \\ m(0) = m_0 & \text{in } \Omega. \\ + BC \end{cases} \tag{2}$$

*admits at most one weak sol. $m \in L^1(Q_T)_+$: $m|b|^2 \in L^1(Q_T)$.*

*Moreover, in this case any weak solution is a renormalized solution, belongs to $C^0([0,T];L^1)$ and satisfies (for a suitable truncation $T_k(\cdot)$):*

$$(T_k(m))_t - \Delta T_k(m) - \mathrm{div}(T_k'(m) m\, b) = \omega_k\,, \qquad \text{in } Q_T \tag{3}$$

*where $\omega_k \in L^1(Q_T)$, and $\omega_k \overset{k\to\infty}{\to} 0$ in $L^1(Q_T)$.*

Main idea: *a nonlinear look at a linear equation*

- If $m|b|^2 \in L^1$, then

$$m = \lim_\varepsilon m^\varepsilon \,,$$

$$\begin{cases} m^\varepsilon_t - \Delta m^\varepsilon - \mathrm{div}(\sqrt{m^\varepsilon}\, B^\varepsilon) = 0\,, & \text{in } (0,T)\times\Omega, \\ m^\varepsilon(0) = m_0\,, \ + \text{BC} & \end{cases}$$

where $B^\varepsilon \stackrel{L^2}{\to} \sqrt{m}\, b$.

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where $B^\varepsilon \xrightarrow{L^2} \sqrt{m}\, b$.

- for general convection-diffusion problems (possibly nonlinear)

$$\begin{cases} m^\varepsilon_t + A m^\varepsilon = \; \mathrm{div}\,(\phi(t,x,m^\varepsilon)) & \text{in } Q_T \\ m^\varepsilon(0) = m_0^\varepsilon\,, \; +\text{BC} \end{cases}$$

we have that if

$$|\phi(t,x,m)| \le c(1+\sqrt{m})\, k(t,x)\,, \qquad k \in L^2(Q_T) \tag{4}$$

then

$$m^\varepsilon \to m \quad \text{in } C^0([0,T];L^1)$$

and $m$ is renormalized solution relative to $m_0$.

- One can apply this idea even in the Di Perna-Lions approach, regularizing $m$ by convolution:

$$m_t - \Delta m - \mathrm{div}(m\,b) = 0 \qquad \star\, \rho_\varepsilon$$

$$\Rightarrow m^\varepsilon := m \star \rho_\varepsilon \quad \text{solves}$$

$$m_t^\varepsilon - \Delta m_\varepsilon - \mathrm{div}((mb) \star \rho_\varepsilon) = 0$$

where Schwartz's inequality $+ \; m \geq 0$ imply

$$|(mb) \star \rho_\varepsilon| \leq \underbrace{(m \star \rho_\varepsilon)^{\frac{1}{2}}}_{\sqrt{m^\varepsilon}} \underbrace{\left((m|b|^2) \star \rho_\varepsilon\right)^{\frac{1}{2}}}_{B^\varepsilon}$$

with $B^\varepsilon$ converging in $L^2(Q_T)$.

$\to$ for purely second order operators, no need of commutators !

Summary on FP:

- the class of weak solutions $m$ such that $m|b|^2 \in L^1$ gives: uniqueness, renormalized formulation, solutions obtained by regularization, estimates. Ex:

$$m_0 > 0\,, \log m_0 \in L^1_{loc}(\Omega) \Rightarrow \log m(t) \in L^1_{loc}(\Omega),$$

hence $m(t) > 0$ a.e.

- the class $m|b|^2 \in L^1$ is consistent with the stochastic flow:

we are considering only trajectories $X_t$ along which the drift is $L^2$-integrable:

$$\mathbb{E}\left[\int_0^T |b(X_t)|^2\right] dt < \infty$$

Work in progress: prove that uniqueness in law holds for (SDE) under this condition and establish rigorously the connection with a stochastic flow

$$\begin{cases} -u_t - \Delta u + H(x, \nabla u) = F(x, m)\,, \\ m_t - \Delta m - \text{ div } (m\, H_p(x, \nabla u)) = 0\,, \\ u(T) = G(x, m(T))\,,\ m(0) = m_0 \end{cases}$$

- $F, G \in C^0(\overline{\Omega} \times \mathbb{R})$
- $p \mapsto H(x, p)$ is convex and satisfies structure conditions

Ex: $H \simeq \gamma(t, x)|\nabla u|^q$, $q \leq 2$.

**Def. of weak solutions**:

- $u, m \in C^0([0, T]; L^1)$, $m\,|Du|^q \in L^1$

-$G(x, m(T)) \in L^1$, $H(x, Du) \in L^1$, $F(x, m) \in L^1$,

- the equations hold in the sense of distributions.

## Theorem (P., ARMA '15)

*Assume that* $m \mapsto G(x,m)$ *is nondecreasing, and let* $m_0 \in L^\infty_+$*.*

*(i) If* $F$*,* $G$ *are bounded below, then there exists a weak solution.*

*(ii) If in addition* $m \mapsto F(x,m)$ *is nondecreasing,* $p \mapsto H(x,p)$ *is strictly convex (at infinity), and* $\log m_0 \in L^1_{loc}(\Omega)$*, then there is a unique weak solution.*

Rmk: The coupling functions $F, G$ have no growth restriction from above

- The case $F = F(x)$ is included !! New viewpoint for

$$\begin{cases} u_t - \Delta u + H(x, Du) = F(x) \\ u_{\partial\Omega} = 0\,, \ \ u(0) = u_0 \end{cases}$$

Uniqueness $\iff$ $m_t - \Delta m - \operatorname{div}(H_p(x,Du)\, m) = 0$ admits a sol. $m$ with $H_p(Du) \in L^2(m)$.

# Convergence of numerical schemes

[Achdou-P., SINUM 2016]

We use finite differences approximations of the mean field games system as in [Achdou-Capuzzo Dolcetta], [Achdou-Camilli-Capuzzo Dolcetta]:

$$\begin{cases} \frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} - (\Delta_h u^k)_{i,j} + g(x_{i,j}, \left[\nabla_h u^k\right]_{i,j}) = F(m_{i,j}^{k+1}), \\ \frac{m_{i,j}^{k+1}-m_{i,j}^{k}}{\Delta t} - (\Delta_h m^{k+1})_{i,j} + \mathcal{T}_{i,j}(u^k, m^{k+1}) = 0, \end{cases}$$

where $g$ is a monotone approximation of the Hamiltonian $H$ as in upwind schemes:
Ex (1-d): $g = g\left(\frac{u_{i+1}-u_i}{h}, \frac{u_i - u_{i-1}}{h}\right)$ with $g(p_1, p_2)$ increasing in $p_2$ and decreasing in $p_1$, $g(q,q) = H(q)$.

while $\mathcal{T}$ is the discrete adjoint of the associated linearized transport:

$$\mathcal{T}(v,m)\cdot w = m\, g_p([\nabla_h v])\cdot[\nabla_h w]$$

This structure allows us to have discrete estimates and compactness as in the continuous model.

For simplicity, assume dimension $N = 2$.

Let $u_{h,\Delta t}$ and $m_{h,\Delta t}$ be the piecewise constant functions respectively taking the values $u_{i,j}^{n+1}$ and $m_{i,j}^{n}$ in $(t_n, t_{n+1}) \times (ih - h/2, ih + h/2) \times (jh - h/2, jh + h/2)$.

**Theorem (Achdou-P. 2015)**

*Up to the extraction of a subsequence of $h$ and $\Delta t$, $m_{h,\Delta t} \to \tilde{m}$ and $u_{h,\Delta t} \to \tilde{u}$ in $L^\beta((0,T)\times\Omega)$ for all $\beta \in [1,2)$, where $\tilde{u}, \tilde{m} \in C^0([0,T]; L^1)$ is a weak solution of the system.*

Rmks:

- Provides a new proof for the existence of weak solutions.
- Uses discrete energy estimates for the Fokker-Planck equation + compactness through discrete Aubin-Simon lemma
- Uses new tools for discrete approximation of viscous HJ with $L^1$-data (monotonicity of the discrete Hamiltonian is crucial)

# Extensions, work in progress

- Similar results exist for the case of $\mathbb{R}^N$. If $m_0 \in L^1 \cap L^\infty$, existence and uniqueness of solutions $(u, m)$ such that $m|Du|^q \in L^1$ and $u \in L^\infty + L^\infty((0, T); L^1)$ (global integrability).

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- [Achdou-P.] Extension to congestion models in mean field games:
$$H = \frac{|Du|^q}{(c + m^\gamma)}, \qquad c \geq 0\,.$$
This corresponds to a cost which penalizes the areas at high density: $L(m, \alpha) \simeq m^{\gamma/q-1}\, |\alpha|^{q'}$.

  We have existence and uniqueness of weak solutions for $0 < \gamma \leq 4\frac{q-1}{q}$ (optimal range).

  Important: the structure of optimality system is missing and optimal control approach cannot be used. Yet the weak theory works...

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- (in progress) Verification theorems for weak solutions
  $\rightarrow$ build trajectories of the SDE associated to weak solutions:

$$dX_t = b(X_t)dt + \sqrt{2}dB_t$$

and prove uniqueness in law for martingale solutions

# Vanishing viscosity and first order case

[joint work with Cardaliaguet, Graber & Tonon]

$$\begin{cases} -\partial_t u + H(t,x,Du) = F(t,x,m) & \text{in } (0,T)\times\Omega \\ \partial_t m - \mathrm{div}(m\, H_p(t,x,Du)) = 0 & \text{in } (0,T)\times\Omega\,, \\ u(T) = u_T\,,\ \ m(0) = m_0 \end{cases}$$

where $u_T, m_0$ are regular.

Structure conditions:

$$c_0\,(m^\gamma - 1) \le F(t,x,m) \le C_0(m^\gamma + 1) \qquad \gamma > 0$$

$$c_1\,(|\xi|^q - 1) \le H(t,x,\xi) \le C_1(|\xi|^q + 1) \qquad p > 1$$

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Energy estimates $\Rightarrow$ $\quad m|Du|^q \in L^1,\ F(m)m \in L^1$

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Energy estimates $\Rightarrow$ $\quad m|Du|^q \in L^1,\ F(m)m \in L^1 \simeq m \in L^{\gamma+1}$
$\to F(m) \in L^{\frac{\gamma+1}{\gamma}}$.

Weak solutions of the first order system are defined as:
$u \in W^{1,q}, m \in L^{\gamma+1},\ m|Du|^q \in L^1$ and

(i) $u$ is a distributional subsolution:

$$-u_t + H(x, \nabla u) \leq F(x, m)$$

(ii) $m$ is a distributional solution for the continuity equation

$$m_t - \ \mathrm{div}\ (m\, H_p(x, \nabla u)) = 0$$

(iii) the energy equality holds

$$\int_0^T\int_\Omega m\, F(x, m) dxdt + \int_0^T\int_\Omega m\, \{H_p(x, Du)Du - H(x, Du)\}\, dxdt$$
$$= \int_\Omega m_0\, u(0) - \int_\Omega u_T\, m(T)$$

## Theorem (CGPT '15)

*Assume that $p \mapsto H(x,p)$ is strictly convex and $m \mapsto F(x,m)$ is increasing. Then the first order system*

$$\begin{cases} -u_t + H(x, \nabla u) = F(x,m)\,, \\ m_t - \ div\,(m\, H_\xi(x, \nabla u)) = 0\,, \\ m(0) = m_0\,,\ u(T) = u_T \end{cases}$$

*admits a unique weak solution $(u,m)$ in the sense that m is unique and $u$ is unique in $\{m > 0\}$.*

- Something more is known if $\frac{\gamma+1}{\gamma} > 1 + \frac{N}{p}$ [Cardaliaguet-Graber] $u$ is Hölder continuous and is also globally unique (not only in $\{m > 0\}$) if one requires

$$-\partial_t u + H(x, Du) \geq 0 \qquad \text{in } Q_T \tag{5}$$

  in the viscosity sense.
- The result extends to second order degenerate case $-\mathrm{Tr}(A(x)D^2(\cdot))$ provided $A$ is Lipschitz and $q \geq \frac{\gamma+1}{\gamma}$.

Sketch of proof: vanishing viscosity limit

$$\begin{cases} -\partial_t u - \varepsilon\Delta u + H(t,x,Du) = F(t,x,m) & \text{in } (0,T)\times\Omega \\ \partial_t m - \varepsilon\Delta m - \mathrm{div}(m\, H_p(t,x,Du)) = 0 & \text{in } (0,T)\times\Omega \end{cases}$$

- Energy estimates give $F(m^\varepsilon)$ bounded in $L^r$, $r = \frac{\gamma+1}{\gamma}$.
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  but the energy equality also holds:

$$\int_0^T\!\!\int_\Omega m\,\alpha + \int_0^T\!\!\int_\Omega \{w\cdot Du - m\,H(x,Du)\} = \int_\Omega m_0\, u(0) - \int_\Omega u_T\, m(T)$$

- By Minty's argument we identify $w = mH_p(Du)$, $\alpha = F(m)$.

# Integral estimates for HJ

Assume $H$ coercive & superlinear : $H(t,x,Du) \geq \alpha |Du|^q$, with $q > 1$.
This leads to estimates for distributional subsolutions

$$\begin{cases} -\partial_t u + c_0 \, |Du|^q \leq f \in L^r(Q_T) \\ u(T) = u_T \in L^\infty \end{cases}$$

Then we have, if $N =$ space dimension:

$$r > 1 + \frac{N}{q} \Rightarrow u \in L^\infty(Q_T)$$

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Qn: where does the exponent come from ?

If we knew that $\partial_t u, |Du|^q \in L^r$ - same space as $f$ - then apply Sobolev embedding + space-time interpolation (a la Gagliardo-Nirenberg)...

Qn: maximal regularity for (coercive) first order problems ?

$$\partial_t u + |Du|^q = f \in L^r(Q_T) \quad \Rightarrow \quad |Du|^q, \partial_t u \in L^r \quad ??$$

**Case $\mathbf{r > 1 + \frac{N}{q}}$** (for solutions)

$$\begin{cases} -\partial_t u + H(t,x,Du) = f \quad \in L^r(Q_T) \\ \alpha\,|Du|^q - c \le H(t,x,Du) \le \beta |Du|^q + c \end{cases}$$

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[Cardaliaguet-P.-Tonon '15] $\Rightarrow u \in W^{1,1}_{loc}(Q_T)$ and
(i) there exists $\varepsilon > 0$ : $|Du|^q, u_t \in L^{1+\varepsilon}_{loc}(Q_T)$ and

$$\|u_t\|_{L^{1+\varepsilon}(K)} + \||Du|^q\|_{L^{1+\varepsilon}(K)} \le C\left(K, \|f\|_{L^r}, \|Du\|_q, r, N, q\right))$$

(ii) $u$ is a.e. differentiable in $Q_T$.

- $\varepsilon$ depends on $\alpha, \beta, N, q, r$ and cannot be arbitrarily large (counterexample !.. in which the conclusion is false for $\varepsilon \ge \frac{1}{q}$)
- This is a Meyers' type result: measurable coefficients, gain of integrability

- Compare with elliptic-parabolic regularity:

$$\partial_t u - \mathrm{div}(A(x)Du) = \mathrm{div}(G)\,, \qquad G \in L^r \;\; r > 2$$

(i) [Meyers] $\rightarrow$ If $A(x)$ elliptic with bounded measurable entries then $\exists\ \varepsilon > 0$: $Du \in L^{2(1+\varepsilon)}$

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Is something similar happening for first order equations ?

$$\partial_t u + a(t,x)|Du|^q = f\,, \qquad f \in L^r \;\; r > 1 + \frac{N}{q}$$

(i) [CPT] : $a(t,x)$ coercive and bounded, only measurable $\rightarrow \quad Du \in L^{q(1+\varepsilon)}$

(ii) $a(t,x)$ coercive and continuous (smooth as needed...) $\Rightarrow$ maximal regularity ?

Thanks for the attention !