Lipschitz regularity results for nonlinear strictly elliptic equations and applications

Vinh Duc NGUYEN

Cardiff university, United Kingdom.
Based on joint works with O.LEY (Rennes)

Final conference of ANR Hamilton-Jacobi equations on networks project.
Rennes May 30 - June 3, 2016.

Problems

We are interested in the gradient bound $||D\phi^\epsilon||_\infty$, **uniform in** ε and $||Du(.,t)||_\infty$ **uniform in** t, for solutions of

[Approx eqn] $\epsilon\phi^\epsilon(x) - \mathrm{trace}[A(x)D^2\phi^\varepsilon(x)] + H(x, D\phi^\epsilon(x)) = 0$

and

[Parabolic eqn] $$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2u) + H(x, Du) = 0 \\ u(x,0) = u_0(x). \end{cases}$$

in periodic setting ($\mathbb{T}^N := \mathbb{R}^N/\mathbb{Z}^N$) and when the diffusion matrix is strictly elliptic, i.e., there exists $\nu > 0$ such that $A(x) \geq \nu I, \quad x \in \mathbb{T}^N$.
In some results, we only need degenerate diffusion, i.e., $\nu = 0$ and we will write $A \geq 0$.

Problems

In this talk, we present the results about the gradient bound $||D\phi^{\epsilon}||_{\infty}$ **uniform in** ε for solutions of

[Approx eqn]$\epsilon\phi^{\epsilon}(x) - \mathrm{trace}[A(x)D^2\phi^{\varepsilon}(x)] + H(x, D\phi^{\epsilon}(x)) = 0$

where

$* : A(x) \geq \nu I$

Terminologies
$*$: $A(x)$: **Ellipticity**
$*$: $H(x,p)$: **Hamiltonian**

General approaches to get lipschitz regularity

$*$ strictly elliptic equations : **Ellipticity** $\Rightarrow$ **lipschitzity :** .
$\longmapsto$ Need suitable growth power in gradient variable (subquadratic for instance, to be dominated by ellipticity).
$*$ degenerate elliptic equations : **Hamiltonian** $\Rightarrow$ **lipschitzity**.
$\longmapsto$ Hence the Hamiltonian have to satisfy some convex-type structures which allow to use the Bernstein's method.

• We present in this talk two results which complement each other toapplied for **any growth power** in gradient variable.
• The assumptions are quite natural and having **no relations with convex-type structures**.

Getting Lipschitzity from Ellipticity 1

Let Ω be a bounded set in $\mathbb{R}^N$.
Ishii-Lions-90 : Assume there are constants $\alpha > 0$, C such that $|H(x,p) - H(y,p)| \leq w(|x-y|)|x-y|^\alpha |p|^{\alpha+2} + C$ for all $x, y \in \Omega, p \in \mathbb{R}^N$.
Barles-91 Assume there are constants $\alpha > 0$, C such that for all $x, y \in \Omega, p \in \mathbb{R}^N$
$|H(x,p) - H(y,p)| \leq C|x-y||p|^3 + C(1+|p|^2)$.

In bounded sets, the lipschitz bound is permitted to depend on the L^∞ norm of solution and this dependence permits us to use **localization arguments**.

Getting Lipschitzity from Ellipticity 2

In the context :

[Approx eqn]$\epsilon\phi^\epsilon(x) - \mathrm{trace}[A(x)D^2\phi^\varepsilon(x)] + H(x, D\phi^\epsilon(x)) = 0,$

In Barles-Souganidis 01 the authors require the existence of a function $\chi : [0, \infty) \to \mathbb{R}$ such that

$$\int_0^\infty \frac{1}{\chi(x)} dx = +\infty, \quad |H(x,p)| \le \chi(|p|).$$

One can see that typically $|H(x,p)| \le C(|p|+1)$, which is much less general than the previous ones.

The reason is that $||\phi^\varepsilon||_\infty$ is not uniformly bounded in ε in general. So **localization arguments cannot be used.**

Oscillation bound for elliptic equation

$\rightarrow$ We now can use **localization arguments** by proving the following uniform oscillation bound

Ley-Nguyen (Non.Analy 2015). Assume $A \geq 0$ and $\frac{H(x,p)}{|p|}$ is coercive, there exists $K < +\infty$ independent of ε s.t

$$\operatorname{osc}(\phi^\varepsilon) := \sup_{\mathbb{T}^N} \phi^\varepsilon - \inf_{\mathbb{T}^N} \phi^\varepsilon \leq K.$$

Oscillation bound for parabolic equation

Suppose that $A \geq 0$, $\frac{H(x,p)}{|p|}$ is coercive and the comparison principle holds for

$$\textbf{[Parabolic \quad eqn]} \frac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2 u) + H(x, Du) = 0.$$

Then there exists $K < +\infty$ independent of t such that

$$u(x,t) - u(y_t, t) \leq L|x - y_t|, \qquad \text{for all } t \geq 0,\ x \in \mathbb{T}^N$$

and y_t such that $u(y_t, t) = \min_{x \in \mathbb{T}^N} u(x,t)$.

First result

subquadratic Hamiltonians Assume there are constants $\alpha > 0, \quad C$ such that for all $x, y \in \mathbb{T}^N, p \in \mathbb{R}^N$
$|H(x,p) - H(y,p)| \leq C|x-y|^\alpha |p|^{\alpha+2} + C(1+|p|^2)$.

Assume moreover that $\frac{H(x,p)}{|p|}$ is coercive, then the solution ϕ^ϵ of

[Approx eqn]$\epsilon\phi^\epsilon(x) - \mathrm{trace}[A(x)D^2\phi^\varepsilon(x)] + H(x, D\phi^\epsilon(x)) = 0$

satisfying $||D\phi^\epsilon||_\infty \leq M$, M is independent of ε.

Assumptions

• there exists constants $k > 2, C > 0$ such that

$$[\textcolor{red}{super - quadratic}] H(x,p) \geq \frac{1}{C}|p|^k - C$$

• there exists positive constants α, β with $\alpha + 2 < \beta < (k-1)\alpha + k$ such that for all $x, y \in \mathbb{T}^N, p \in \mathbb{R}^N$,
$|H(x,p) - H(y,p)| \leq C|x-y|^\alpha |p|^\beta + B(p)$
•the growth order of B is less than k, i.e,

$$\frac{B(p)}{|p|^k} \to 0 \quad \text{as } |p| \to +\infty.$$

Assumption on B is natural in view of [super-quadratic]

Some examples

Example 1 Assume that $k > 2, \beta < (k-1)\alpha + k$, then the following Hamiltonian H satisfies our assumptions :

$$H(x,p) = a(x)h(p) + G(x,p),$$

where $0 < a \in C^{0,\alpha}(\mathbb{T}^N)$, $\frac{|p|^k}{C} \leq h(p) \leq C|p|^\beta$ and $\frac{G(x,p)}{|p|^k} \to 0$ as $|p| \to +\infty$.

Example 2 Let K be any continuous function satisfying $\frac{|p|^m}{C} - C \leq K(x,p) \leq C(|p|^M + 1), \quad x \in \mathbb{T}^N, p \in \mathbb{R}^N, m > 2.$

Then, the function

$$H(x,p) = K(x,p) + \alpha|p|^{M+\delta}, \alpha > 0, \delta > 0$$

satisfies our assumptions.

Key ideas 1

Recall the equation under consideration

$$[\textbf{Approx eqn}] \epsilon\phi^\epsilon(x) - \mathrm{trace}[A(x)D^2\phi^\varepsilon(x)] + H(x, D\phi^\epsilon(x)) = 0,$$

The main idea comes from the following observation :
• it seems hard to prove directly a continuous solution is lipschitz.
• it is much easier to prove a Holder continuous solution with a close enough to 1 Holder exponent is lipschitz.

Key ideas 2

The idea turns out to be :

• from Capuzzo Dolcetta, Leoni, and Porretta-2010, the solution is $\frac{k-2}{k-1}$ Holder continuous.

• The oscillation bound permits us to use localization arguments.

• improve this Holder continuous solution to a $\chi > \frac{k-2}{k-1}$ Holder continuous solution, where χ is big enough $(\chi > 1 - \frac{\alpha}{\beta-2})$.

• improve this Holder continuous solution to lipschitz continuous.

With H satisfies one of two contexts and $u_0 \in C^2(\mathbb{T}^N)$, there exists a consant C such that the solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2u) + H(x, Du) = 0, (x,t) \in \mathbb{T}^N \times (0, \infty \\ u(x,0) = u_0(x), \qquad x \in \mathbb{T}^N. \end{cases}$$

satisfies

$$|u(x,t) - u(y,s)| \leq C|x-y| + C|t-s|, \forall x, y \in \mathbb{T}^N, t, s \in [0, \infty)$$

Application to Long Time Behavior of Parabolic equations

With H satisfies one of two contexts, the solution u of

$$\begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{trace}(A(x)D^2u) + H(x, Du) = 0, & (x,t) \in \mathbb{T}^N \times (0, \\ u(x,0) = u_0(x), & x \in \mathbb{T}^N. \end{cases}$$

has the following property

$$u(x,t) + ct \to v(x) \quad \text{uniformly in } \mathbb{T}^N \text{ as } t \to \infty,$$

Assume $A \geq 0$ and

$$\frac{|p|^m}{C} - C \leq H(x,p) \leq C(|p|^M + 1), \quad x \in \mathbb{T}^N, p \in \mathbb{R}^N, m > 2.$$

Question : does there exist a solution of ?

$$u - \mathrm{trace}[A(x)D^2 u] + H(x, Du) = 0, \quad x \in \mathbb{T}^N.$$

can the vanishing viscosity method give the existence of solution of the equation ?

$$u_\varepsilon - \mathrm{trace}[(A(x) + \varepsilon Id)D^2 u_\varepsilon] + H(x, Du_\varepsilon) = 0, \quad x \in \mathbb{T}^N.$$

Idea : We consider

$$u_\varepsilon - \mathrm{trace}[(A(x) + \varepsilon Id)D^2 u_\varepsilon] + H(x, Du_\varepsilon) + \varepsilon |Du_\varepsilon|^{M+1} = 0.$$

$\longmapsto$ The above equation satisfies our second result $\Rightarrow$ the strong comparison principle holds $\Rightarrow$ there exists a continuous solution u_ε of this equation.
$\longmapsto$ Capuzzo Dolcetta, Leoni, and Porretta-2010 yields

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq K|x-y|^{\frac{m-2}{m-1}}, \quad x, y \in \mathbb{T}^N,$$

K does not depend on ε.
$\longmapsto$ Note that $||u_\varepsilon||_\infty$ is bounded uniformly in ε. Ascolli-Arzela theorem yields the existence of a holder continuous solution of the original equation.