Aubry Mather theory for weakly coupled systems of Hamilton-Jacobi equations

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Work in collaboration with Andrea Davini (La Sapienza, Roma 1),
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Plan

References

Related works by

- Camilli-Ley-Loreti-Nguyen,
- Mitake-Tran,
- Cagnetti-Gomes-Mitake-Tran,
- Mitake-Siconolfi-Tran-Yamada,
- Ibrahim-Siconolfi-Zabad
- ...

Plan

Weakly coupled systems of Hamilton-Jacobi equations

Evolution equation

$$\frac{\partial u_i}{\partial t} + H_i(x, D_x u_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(t,x) = 0 \quad \text{in } (0,+\infty) \times \mathbb{T}^N. \qquad \text{(EHJ)}$$

for $i \in \{1, \dots, m\}$, with initial conditions $u_i(0,x) = u_i^0(x)$ where the initial conditions are Lipschitz continuous.

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for $i \in \{1, \ldots, m\}$, with initial conditions $u_i(0,x) = u_i^0(x)$ where the initial conditions are Lipschitz continuous.
In matrix notations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbb{H}(x, D_x \mathbf{u}) + B(x)\mathbf{u} = 0,$$

where $B(x) = \big(b_{ij}(x)\big)_{1 \leqslant i,j \leqslant m}$ and $\mathbb{H}(x, D_x \mathbf{u}) = \big(H_i(x, D_x u_i)\big)_{1 \leqslant i \leqslant m}$.

Weakly coupled systems of Hamilton-Jacobi equations II

Stationary equation

$$H_i(x, D_x u_i) + \sum_{j=1}^{m} b_{ij}(x) u_j(x) = c \quad \text{in } (0, +\infty) \times \mathbb{T}^N. \tag{SHJ}$$

for $i \in \{1, \ldots, m\}$, and some $c \in \mathbb{R}$.

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In matrix notations:

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All considered functions will be at least continuous .
All solutions, subsolutions, supersolutions are meant in the viscosity sense.

The Hypotheses

1. The Hamiltonians :

(H1) $H_i : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$ is continuous ;

(H2) $p \mapsto H_i(x, p)$ is strictly convex on $\mathbb{R}^N$ for any $x \in M$;

(H3) there exist two superlinear functions $\alpha, \beta : \mathbb{R}_+ \to \mathbb{R}$ such that

$$\alpha(|p|) \leqslant H_i(x, p) \leqslant \beta(|p|) \qquad \text{for all } (x, p) \in M \times \mathbb{R}^N.$$

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$$\alpha(|p|) \leqslant H_i(x, p) \leqslant \beta(|p|) \qquad \text{for all } (x, p) \in M \times \mathbb{R}^N.$$

2. The coupling matrix :
   - The function $x \mapsto B(x)$ is continuous ,
   - $b_{ii} \geqslant 0$, $b_{ij} \leqslant 0$ for $j \neq i$, $\sum_{j=1}^{m} b_{ij} \geqslant 0$ for any $i \in \{1, \dots, m\}$.
   - It is degenerate : $\sum_{j=1}^{m} b_{ij} = 0$ for any $i = 1, \dots, m$.
   - $B(x)$ is irreducible : $\forall \mathcal{I} \subsetneq \{1, \dots, m\}, \quad \exists i \in \mathcal{I}, \quad \exists j \notin \mathcal{I}, \quad b_{ij} \neq 0$.

Why those hypotheses?

- The hypotheses on the H_i are standard in weak KAM theory. They allow usually to use variational arguments with the use of the Lax–Oleinik formula, which involves the Lagrangian.

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- First two hypotheses on B allow to obtain a comparison principle (Engler-Lenhart 91 on monotonous systems, Camilli-Ley-Loreti 10):

Theorem

Let $\mathbf{u}_0$ be a Lipschitz initial data and $\mathbf{w}, \mathbf{v} : \left([0, T) \times \mathbb{T}^N\right)^m$ be a subsolution and a supersolution of EHJ, then $\mathbf{w} \leqslant \mathbf{v}$.
In particular there exists a unique solution $\mathbf{u}$ to EHJ.

We denote $S(t)\mathbf{u}_0 = \mathbf{u}(t, \cdot)$.

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- Irreducibility equivalent to for all i, j there is n such that $B^n_{ij} \neq 0$: the system cannot be split, all equations communicate. Ker $B = \mathbb{R}\mathbb{1}$.

Irreducibility implies a priori compactness :

Proposition

Let $c \in \mathbb{R}$, there exists a constant K such that if $\mathbf{u}$ verifies $\mathbb{H}(x, D_x\mathbf{u}) + B(x)\mathbf{u} \leqslant c\mathbb{1}$, then each u_i is K–Lipschitz and $\max u_i(x) - \min u_j(y) \leqslant K$.

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There exists a unique constant c_0 such that the stationary equation $\mathbb{H}(x, D_x\mathbf{u}) + B(x)\mathbf{u} = c_0\mathbb{1}$ admits solutions.
Solutions to this equation are called weak KAM solutions and the constant c_0 is the critical constant .

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- In particular, $S(t)\mathbf{u}$ is a critical subsolution for all $t > 0$.

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1. if $x \in \mathcal{A}$, any subsolution u is differentiable at x and $H(x, D_x u) = c_0$.
2. (Fathi-Siconolfi) there exists a C^1 subsolution u such that $H(x, D_x u) < c_0$ if $x \notin \mathcal{A}$.
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Goal: recover those results for systems (no dynamical or variational tools. Only PDE methods) .

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The set $\mathcal{S}$ is closed and convex .

The Aubry set

Theorem (Davini–Z.)

There exists a non–empty closed set $\mathcal{A} \subset \mathbb{T}^N$ *such that:*

1. *for all* $\mathbf{u} \in \mathcal{S}$ *and* $t > 0$, $S(t)\mathbf{u}(x) + tc_0 = \mathbf{u}(x)$.
2. *there exists a subsolution* $\tilde{\mathbf{u}}$, C^1 *on* $\mathbb{T}^N \setminus \mathcal{A}$ *such that*

$$\forall x \in \mathbb{T}^N \setminus \mathcal{A}, \quad \mathbb{H}(x, D_x\tilde{\mathbf{u}}) + B(x)\tilde{\mathbf{u}} < c_0\mathbb{1}.$$

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Open question

Does there exist a C^1 subsolution?

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It entails some rigidity for the critical equation:

Proposition

1. *if* $\mathbf{u}$ *and* $\mathbf{v}$ *are in* $\mathcal{S}$ *and* $x \in \mathcal{A}$, *then* $(\mathbf{u} - \mathbf{v})(x) \in \mathbb{R}\mathbb{1}$.

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Open question

Are subsolutions differentiable on the Aubry set?

A motivating example à la Namah–Roquejoffre (Camilli, Ley, Loreti, N'guyen)

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2. $\mathcal{A} = \cap f_i^{-1}\{0\}$,
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Theorem (Camilli, Ley, Loreti, N'guyen)

For any $\mathbf{u}_0$, the solution $S(t)\mathbf{u}_0$ to EHJ converges to a weak KAM solution as $t \to +\infty$.

Plan

1. PDE aspects
2. Dynamical aspects

Classical Aubry Mather Theory

Define $L(x, v) = \sup_p \langle p, v\rangle - H(x, p)$ then if $\gamma : [0, t] \to \mathbb{T}^N$ is a loop ,

$$\int_0^t L(\gamma, \dot\gamma) \geqslant -tc_0.$$

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A point $x \in \mathcal{A}$ if and only if there are loops $\gamma_n : [0, t_n] \to \mathbb{T}^N$ with $\gamma_n(0) = \gamma_n(t_n) = x$ and $t_n \geqslant 1$ such that

$$\int_0^{t_n} L(\gamma_n, \dot\gamma_n) + t_n c_0 \to 0.$$

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- Let $\mathcal{D}$ be the space of càdlàg paths $\omega : [0, +\infty) \to \{1, \ldots, m\}$.
- Let $\mathcal{D}(\mathbb{R}^N)$ be the space of càdlàg paths $\omega : [0, +\infty) \to \mathbb{R}^N$.

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- Let $\mathcal{D}(\mathbb{R}^N)$ be the space of càdlàg paths $\omega : [0, +\infty) \to \mathbb{R}^N$.
- There exists a Probability measure $\mathbb{P}$ on $\mathcal{D}$ such that for $t, h > 0$, $i \neq j$,

$$\mathbb{P}(\omega(t+h) = j \mid \omega(t) = i) = -hb_{ij} + o(h).$$

Stopping times and admissible strategies

- A stopping time $\tau : \mathcal{D} \to [0, +\infty)$ is a measurable function (random variable) adapted to the natural filtration on $\mathcal{D}$.
  Roughly speaking: if $\omega^1_{[0,T]} = \omega^2_{[0,T]}$ and $\tau(\omega^1) < T$ then $\tau(\omega^1) = \tau(\omega^2)$.

- An admissible strategy is a random variable : $\Xi : \mathcal{D} \to \mathcal{D}(\mathbb{R}^N)$ which is
  - locally (in time) bounded: for all $t > 0$ there is $R > 0$ such that $\Xi(\omega)(s) \leqslant R$, for a.e. ω and $s \leqslant t$.
  - adapted to the natural filtration on $\mathcal{D}$ or non anticipating . This means that $\omega^1_{[0,T]} = \omega^2_{[0,T]}$ implies $\Xi(\omega^1)_{[0,T]} = \Xi(\omega^2)_{[0,T]}$.

Trajectories

Given an admissible strategy , define its random trajectory by

$$\mathcal{I}(\Xi, \omega, t) = \int_0^t \Xi(\omega, s)\mathrm{d}s.$$

If τ is a bounded stopping time and $x \in \mathbb{T}^N$, define $\mathcal{K}(\tau, x)$ as the set of trajectories reaching x at τ meaning

$$\forall \omega \in \mathcal{D}, \quad \mathcal{I}\big(\Xi, \omega, \tau(\omega)\big) = x.$$

Lax–Oleinik type caracterization of subsolutions (Mitake, Siconolfi, Tran, Yamada)

Define $L_i(x,v) = \sup_p \langle p, v\rangle - H_i(x,p)$ then

Theorem

1. *The function* $\mathbf{u}$ *is a subsolution if and only if for all* $x, y \in \mathbb{T}^N$, $i \in \{1, \dots, m\}$ *bounded stopping time* τ *and admissible strategy* $\Xi \in \mathcal{K}(\tau, y - x)$

$$u_i(x) - \mathbb{E}_i\big(u_{\omega(\tau)}(y)\big) \leqslant \mathbb{E}_i\Big[\int_0^{\tau(\omega)} L_{\omega(s)}\big(x + \mathcal{I}(\Xi)(s), -\Xi(s)\big) + c_0 ds\Big].$$

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2. *Given* $\boldsymbol{b} \in \mathbb{R}^m$ *and* $x \in \mathbb{T}^N$, *there exists a subsolution such that* $\mathbf{u}(x) = \boldsymbol{b}$ *if and only if for all* $i \in \{1,\dots,m\}$, τ *bounded stopping time and* $\Xi \in \mathcal{K}(\tau, 0)$ *admissible strategy*

$$\mathbb{E}_i\Big[\int_0^{\tau(\omega)} L_{\omega(s)}\big(x + \mathcal{I}(\Xi)(s), -\Xi(s)\big) + c_0 ds - b_i + b_{\omega(\tau)}\Big] \geqslant 0.$$

Dynamical caracterization of the Aubry set (Ibrahim, Siconolfi, Zabad)

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$$\inf_{\substack{\tau \gg 1 \\ \Xi \in \mathcal{K}(\tau,0)}} \mathbb{E}_i \Big[\int_0^{\tau(\omega)} L_{\omega(s)}\big(x + \mathcal{I}(\Xi)(s), -\Xi(s)\big) + c_0 ds - b_i + b_{\omega(\tau)} \Big] = 0$$

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3. *The following relation holds for some $\boldsymbol{b} \in \mathbb{R}^m$ and all $i \in \{1, \dots, m\}$:*

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Lax–Oleinik for EHJ (Davini, Siconolfi, Z.)

Theorem

Let $\mathbf{u}^0$ be a Lipschitz initial data, and $t > 0$, $i \in \{1, \ldots, m\}$, then

$$\big(S(t)\mathbf{u}^0\big)_i(x) = \min_{\Xi} \mathbb{E}_i\Big[u_{\omega(t)}\big(\mathcal{I}(\Xi, \omega, t)\big) + \int_0^t L_{\omega(s)}\big(x + \mathcal{I}(\Xi)(s), -\Xi(s)\big)\mathrm{d}s\Big].$$

Idea of proof I: inequality

Let Ξ be an admissible strategy, $t, h > 0$. Write $f(t) = \mathbb{E}_i\Big[u_{\omega(t)}\big(\mathcal{I}(\Xi,\omega,t)\big)\Big]$. Write

$$\frac{f(t+h)-f(t)}{h} = \mathbb{E}_i\big(\psi_h(\omega)\big) + \mathbb{E}_i\big(\phi_h(\omega)\big)$$

where

$$\psi_h(\omega) := \frac{u_{\omega(t+h)}\big(t,\mathcal{I}(\Xi,\omega,t)\big) - u_{\omega(t)}\big(t,\mathcal{I}(\Xi,\omega,t)\big)}{h}$$

$$\varphi_h(\omega) := \frac{u_{\omega(t+h)}\big(t+h,\mathcal{I}(\Xi,\omega,t+h)\big) - u_{\omega(t+h)}\big(t,\mathcal{I}(\Xi,\omega,t)\big)}{h}.$$

Idea of proof II: inequality

Compute that

$$\mathbb{E}_i\big(\psi_h(\omega)\big) \to -\mathbb{E}_i\big[(B\mathbf{u})_{\omega(t)}\big(t, \mathcal{I}(\Xi, \omega, t)\big)\big].$$

and that (when well defined)

$$\mathbb{E}_i\big(\phi_h(\omega)\big) \to \mathbb{E}_i\big[\partial_t u_{\omega(t)} + D_x u_{\omega(t)} \cdot \Xi(\omega, t)\big].$$

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Combined with the Fenchel inequality :
$-D_x u_{\omega(t)} \cdot \Xi(\omega, t) \leqslant H_{\omega(t)}(\mathcal{I}(\Xi, \omega, t), D_x u_{\omega(t)}) + L_{\omega(t)}\big(\mathcal{I}(\Xi, \omega, t), -\Xi(\omega, t)\big)$
and by integrating between 0 and t yields

$$\big(S(t)\mathbf{u}^0\big)_i(x) \leqslant \mathbb{E}_i\Big[u_{\omega(t)}\big(\mathcal{I}(\Xi, \omega, t)\big) + \int_0^t L_{\omega(s)}\big(x + \mathcal{I}(\Xi)(s), -\Xi(s)\big)\mathrm{d}s\Big].$$

Idea of proof: equality

For the equality, one needs to find a control such that the Fenchel inequality is an equality, which means that almost everywhere,

$$-\Xi(\omega, t) = \partial_p H_{\omega(t)}(\mathcal{I}(\Xi, \omega, t), D_x u_{\omega(t)})$$

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The optimal Ξ is constructed using the fact that each u_i solves an equation of the form

$$\partial_t u_i + G_i(t, x, D_x u_i) = 0.$$

This should give new insights on known (or not) results such as:

- vanishing discount problem (with Davini),
- long time behavior,
- ...

Thank you for your attention!